Reconstruction of the Scalar Potential and Higgs Self-couplings Measurements at future colliders

Do we expect to learn something from the measurements of the Higgs self-couplings?

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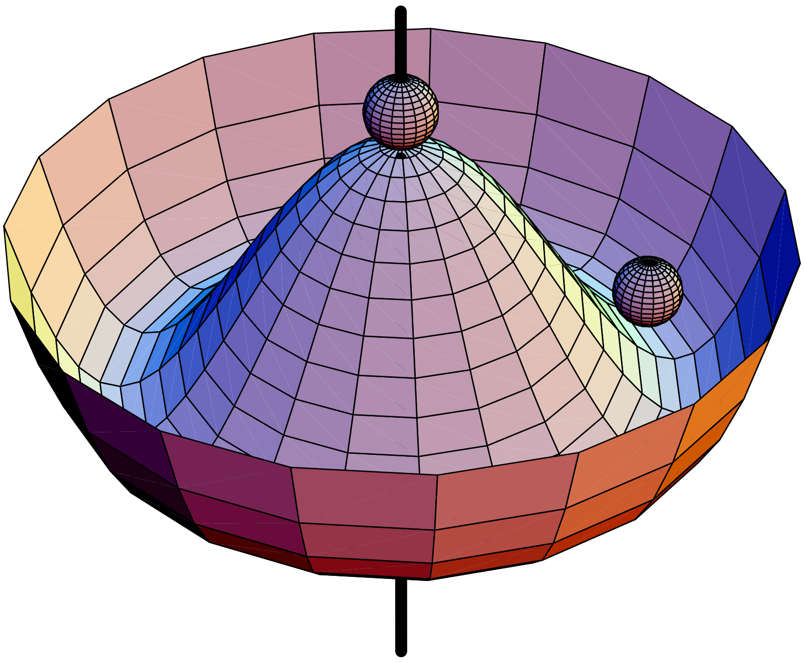


Les Houches: Physics at TeV Colliders III, May 2003

Introduction

- **Talking about the Higgs but where is it?**
- **Higgs searches and discoveries at future machines.**
- **The future first measurements in the Higgs sector.**

"SM", one Higgs doublet

- **Higgs potential and its parameterisation.**
- **The HHH vertex @ the LHC, rates measurements.**
- **The HHH vertex @ the LC, rates measurements.**
- **Isolating the HHH vertex @ the LC, observables.**

SUSY and 2HDM

- **General parameterisation of the 2HDM potential**, leading operators.
- **Hard terms and soft terms, conventional SUSY.**
- **Higgs spectrum, couplings to matter and "decoupling".**
- **Self-couplings and the mass basis.**
- **Screening (the role of no tadpole).**
- **(Radiative Corrections).**
- **Strategies and measurements at e^+e^- .**
- **(higher order operators)**

Conclusions

With the recent discovery of the top, all the matter content of the $\mathcal{SM}$ has been accurately identified.

	SU(3)	$\mathbf{SU(2)}_L$	$\mathbf{U(1)}_Y$	$Q = T_3 + Y$
$Q = (u_L, d_L)$	**3**	**2**	$\frac{1}{6}$	$(\frac{2}{3}, -\frac{1}{3})$
u_R	**3**	**1**	$\frac{2}{3}$	$\frac{2}{3}$
d_R	**3**	**1**	$-\frac{1}{3}$	$-\frac{1}{3}$
$L = (\nu_L, e_L)$	**1**	**2**	$-\frac{1}{2}$	$(0, -1)$
e_R	**1**	**1**	-1	-1
ν_R	**1**	**1**	0	0

All the precision measurements are a glorious confirmation of the gauge principle

$$\begin{aligned} \psi(x) &\rightarrow U(x)\psi(x) = e^{i\theta_3^a(x)T^a} e^{i\theta_2^j(x)\frac{\tau^j}{2}} e^{i\theta_1(x)Y} \psi(x) \ , \\ D_\mu &= \partial_\mu - i g_s T^a A_\mu^a - i g \frac{\tau^i}{2} W_\mu^i - i g' Y B_\mu \ . \end{aligned}$$

The covariant derivatives completely specify the interactions of all known fermions and gauge bosons and encode the universality of the gauge couplings

$$\mathcal{L}_{\text{matter}} = \sum_{j=Q,u_R,d_R,L,e_R,\nu_R} \overline{\psi}_j i\gamma^\mu D_\mu \psi_j \ .$$

An important observation concerning non-Abelian gauge theories is that the gauge fields are self-interacting (confirmed both indirectly and to a good precision, directly at LEP2)

$$W_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\mu W_\nu^i + g\epsilon^{ijk} W_\mu^j W_\nu^k \ .$$

What is missing in the $\mathcal{SM}$ is the

Higgs

In the $\mathcal{SM}$ the Higgs mechanism role is to first give masses to all gauge bosons and fermions in a gauge invariant way. This is introduced via the doublet (only one is needed)

$$\Phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H) \end{pmatrix} e^{i\omega^j \frac{\tau^j}{2v}} \qquad \mathcal{L}_{\rm Higgs} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi^\dagger \Phi)$$

$$\text{with} \qquad V(\Phi^\dagger\Phi) = \lambda \left(\Phi^\dagger \Phi - \frac{v^2}{2} \right)^2 .$$

where H is the physical Higgs field of the electroweak theory, $\omega^{1,2,3}$ the Goldstone bosons and v the vacuum expectation value.

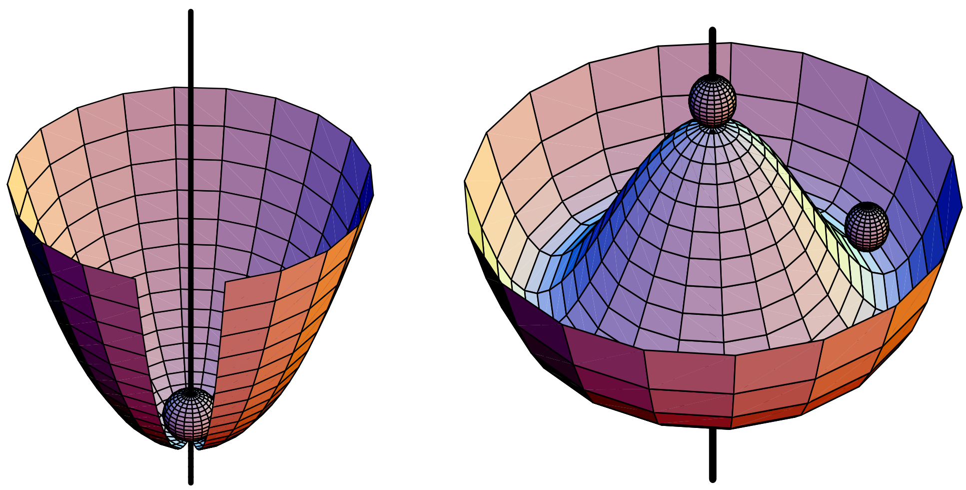
Fermion masses also through Higgs

$$-\mathcal{L}^l_m \;=\; \sum_{i=e,\mu,\tau} y^i_l \left(\bar{L}_i \Phi l_{R,i} + \bar{l}_{R,i} \Phi^\dagger L_i \right) \longrightarrow \text{unitary gauge} \longrightarrow$$

$$\sum_{i=e,\mu,\tau} \frac{y^i_l v}{\sqrt{2}} \left(1 + \frac{H}{v}\right) \bar{l}_i l_i \quad , \quad \frac{y^i_l v}{\sqrt{2}} = m^i_l .$$

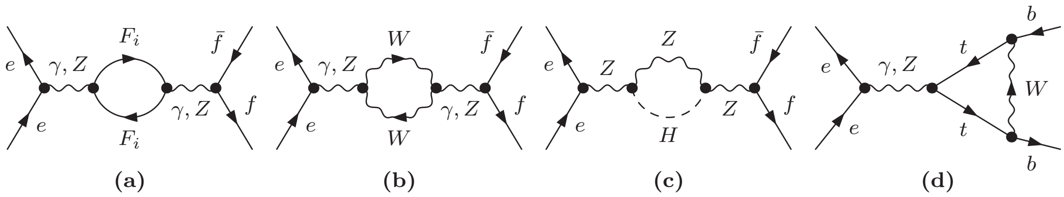
Once the Higgs has been discovered, one needs to measure its coupling to matter, uniquely identified in the $\mathcal{SM}$ and reconstruct the potential

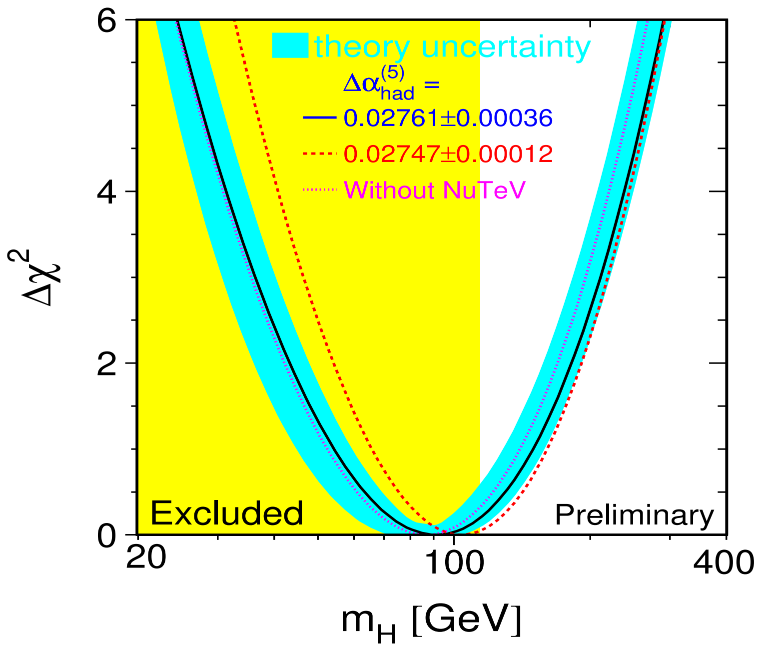
Scalar potentials. The first panel shows a symmetric potential, under rotation around the vertical axis, with a stable minimum where the ball is resting. In this situation $<0|\phi|0>=0$. *The second panel, resembling a mexican hat, illustrates spontaneous symmetry breaking. The symmetric configuration at the top of the hat is unstable. The system will pick up* any *stable configuration along the brim with* $<0|\phi|0>\neq 0$. *The Goldstone mode therefore represents this azimuthal direction whereas the radial component is the Higgs field.*



Where is the Higgs?

From (present) indirect precision loop measurements



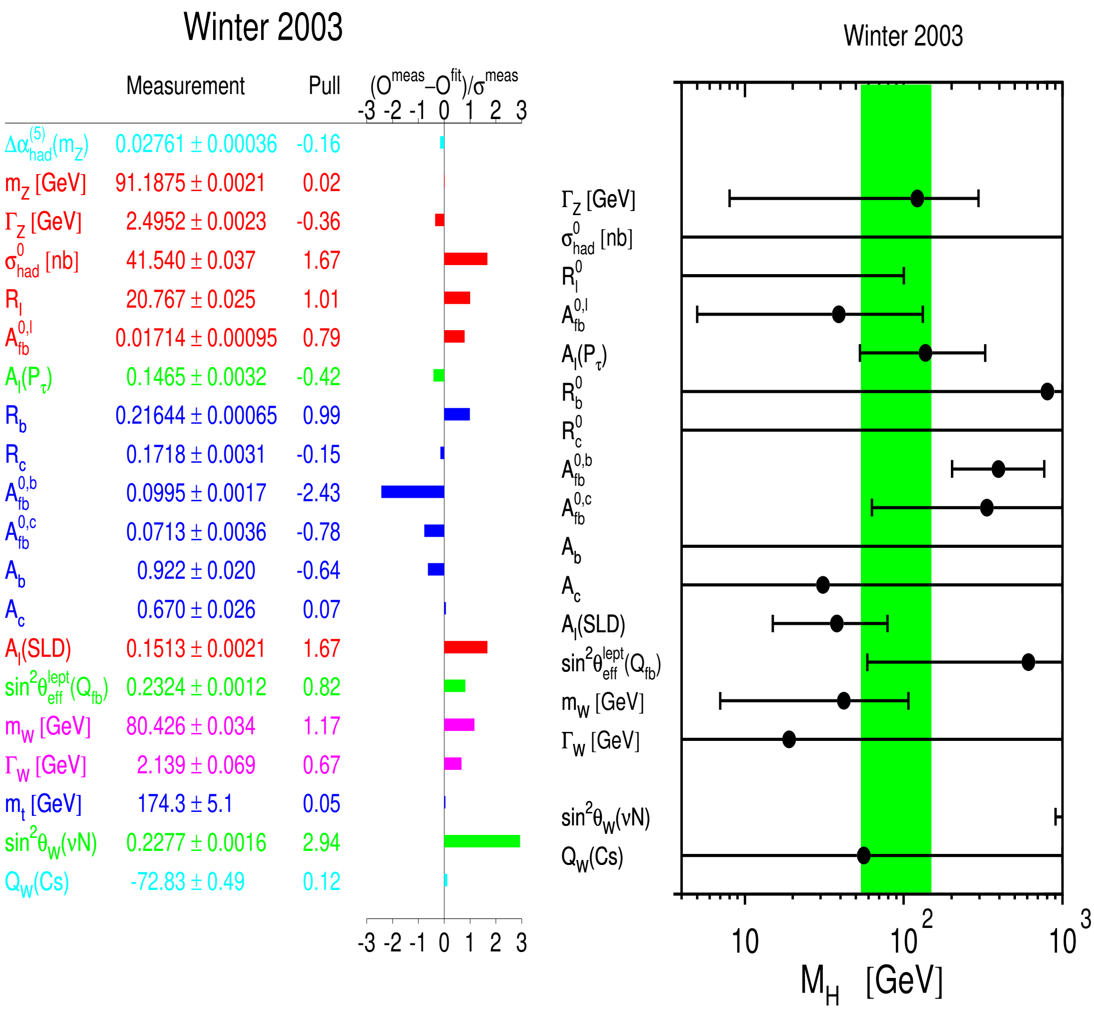


$$M_H = 91^{+58}_{-37}\text{GeV} \quad M_H < 211\text{GeV @95\%CL}$$

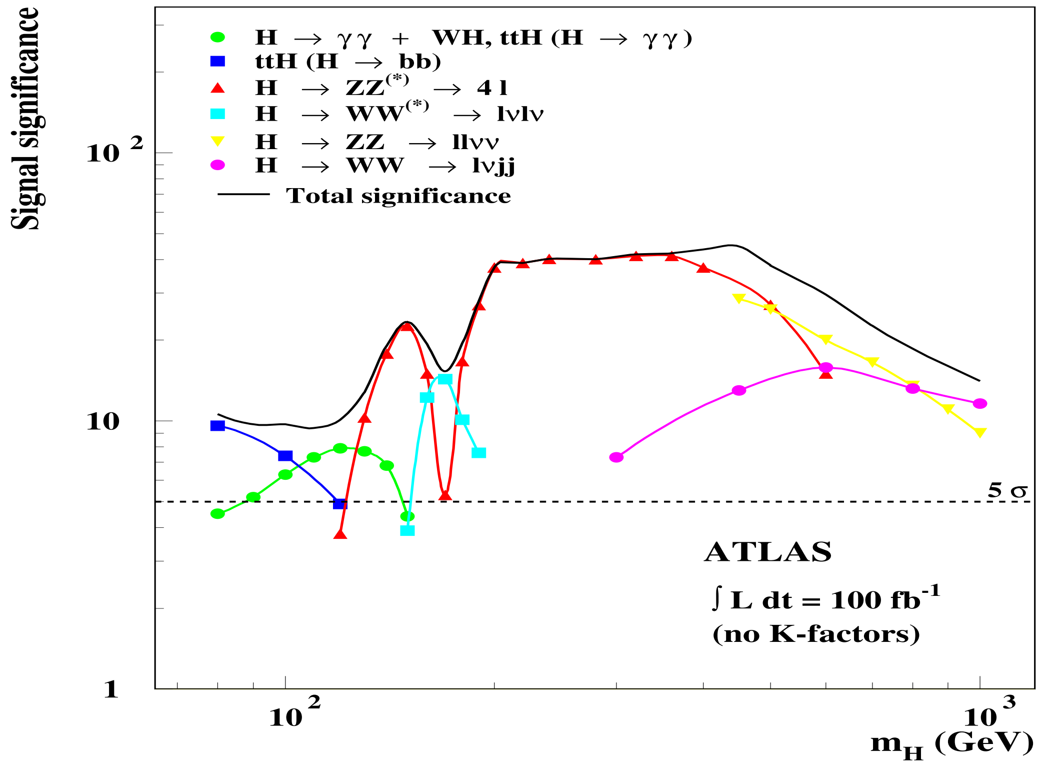
Direct search $M_H > 114.4$GeV @95%CL.

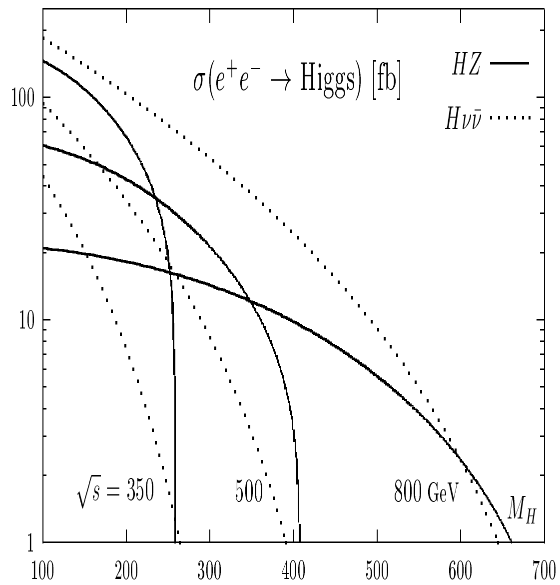
A light Higgs is preferred, with $M_H > 150$GeV perfectly Ok.

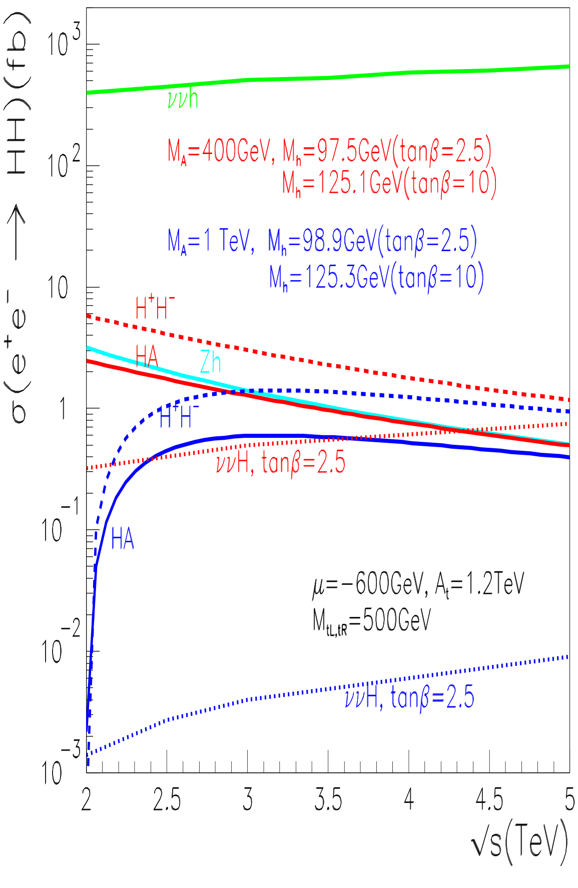
This said it may be that heavier Higgses not completely excluded...


Winter 2003
Measurement
Pull
$(O^{meas}-O^{fit})/\sigma^{meas}$
-3 -2 -1 0 1 2 3
$\Delta\alpha^{(5)}_{had}(m_Z)$ 0.02761 ± 0.00036 -0.16
m_Z [GeV] 91.1875 ± 0.0021 0.02
Γ_Z [GeV] 2.4952 ± 0.0023 -0.36
σ^0_{had} [nb] 41.540 ± 0.037 1.67
R_l 20.767 ± 0.025 1.01
$A^{0,l}_{fb}$ 0.01714 ± 0.00095 0.79
$A_l(P_\tau)$ 0.1465 ± 0.0032 -0.42
R_b 0.21644 ± 0.00065 0.99
R_c 0.1718 ± 0.0031 -0.15
$A^{0,b}_{fb}$ 0.0995 ± 0.0017 -2.43
$A^{0,c}_{fb}$ 0.0713 ± 0.0036 -0.78
A_b 0.922 ± 0.020 -0.64
A_c 0.670 ± 0.026 0.07
A_l(SLD) 0.1513 ± 0.0021 1.67
$\sin^2\theta^{lept}_{eff}(Q_{fb})$ 0.2324 ± 0.0012 0.82
m_W [GeV] 80.426 ± 0.034 1.17
Γ_W [GeV] 2.139 ± 0.069 0.67
m_t [GeV] 174.3 ± 5.1 0.05
$\sin^2\theta_W(\nu N)$ 0.2277 ± 0.0016 2.94
$Q_W(Cs)$ -72.83 ± 0.49 0.12
-3 -2 -1 0 1 2 3
Winter 2003
Γ_Z [GeV]
σ^0_{had} [nb]
R^0_l
$A^{0,l}_{fb}$
$A_l(P_\tau)$
R^0_b
R^0_c
$A^{0,b}_{fb}$
$A^{0,c}_{fb}$
A_b
A_c
A_l(SLD)
$\sin^2\theta^{lept}_{eff}(Q_{fb})$
m_W [GeV]
Γ_W [GeV]
$\sin^2\theta_W(\nu N)$
$Q_W(Cs)$
10 10^2 10^3
M_H [GeV]

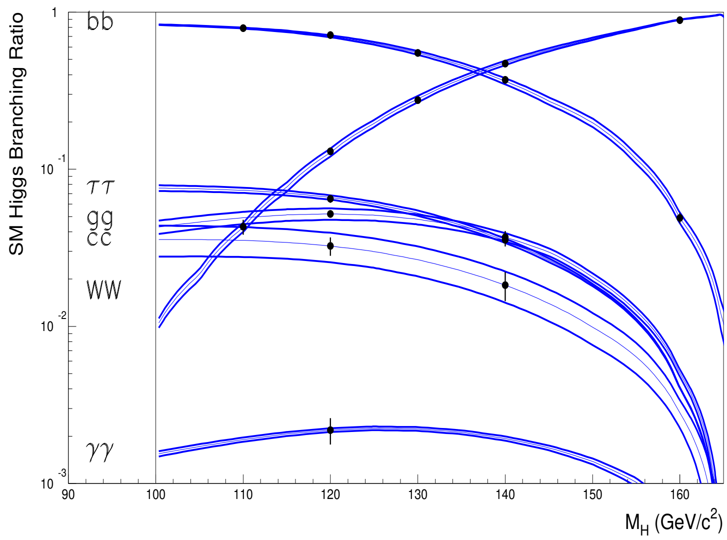
Higgs discovery at LHC and LC







At the next colliders, LHC (SLHC, VLHC), and LC (TESLA, JLC) it has been demonstrated that one should not miss the Higgs. The issue would then be that of precision measurements of its properties: spin, mass, coupling to matter



Summary of the Higgs boson profile from the LHC and the LC data. Relative accuracies for the measurement of Higgs properties for different Higgs boson masses. Battaglia Desch, 2000

	M_H (GeV/c^2)	$\delta(X)/X$ LHC 2×300 fb^{-1}	$\delta(X)/X$ LC 500 fb^{-1}
M_H	120	9×10^{-4}	3×10^{-4}
M_H	160	10×10^{-4}	4×10^{-4}
$\frac{g_{Htt}}{g_{HWW}}$	120	0.070	0.023
$\frac{g_{HZZ}}{g_{HWW}}$	160	0.050	0.022
Γ_{tot}	120-140	-	0.04 - 0.06
g_{Huu}	120-140	-	0.02 - 0.04
g_{Hdd}	120-140	-	0.01 - 0.02
g_{HWW}	120-140	-	0.01 - 0.03
λ_{HHH}	120	-	0.22

Note that only a light Higgs considered!

Clean environment of e^+e^- allows good precision on couplings.

Deviations in the "SM" Higgs potential

$$V_{\mathcal{SSB}} = \lambda \left\{ \sum_{n=2} \frac{g^{2(n-2)}}{\Lambda^{2(n-2)}} \frac{a_n}{(n-1)^2} \left[\Phi^\dagger \Phi - \frac{v^2}{2} \right]^n \right\}$$

(small field expansion of the potential around the vacuum v)

with dim-4 and dim-6 only

$$\begin{aligned} V_{\mathcal{SSB}} &= \frac{1}{2} M_H^2 \left\{ H^2 + \frac{g}{M_W} H (\varphi^+ \varphi^- + \frac{\varphi_3^2}{2}) + \frac{g}{2M_W} h_3 H^3 \right. \\ & \left. h_4 \left(\frac{g}{4M_W} \right)^2 H^4 + h_3' \left(\frac{g}{2M_W} \right)^2 H^2 (\varphi^+ \varphi^- + \frac{\varphi_3^2}{2}) \ldots\ldots \right\} \end{aligned}$$

$$\begin{aligned} h_3 &= 1 + a_3 \frac{M_W^2}{\Lambda^2} = 1 + \delta h_3 \\ h_4 &= 1 + 6 \delta h_3 \\ h_3' &= 1 + 3 \delta h_3 \end{aligned}$$

Other parameterizations of Higgs self-couplings

Barger, Han, Langacker, McElrath, Zerwas, 2003.

$$\mathcal{L}' = \sum_i^2 \frac{f_i}{\Lambda^2}\mathcal{O}_i, \quad \mathcal{O}_1 = \frac{1}{2}\partial_\mu(\Phi^\dagger\Phi)\partial^\mu(\Phi^\dagger\Phi) \quad \text{and} \quad \mathcal{O}_2 = -\frac{1}{3}(\Phi^\dagger\Phi)^3,$$

with the ground state :

$$\langle|\Phi|^2\rangle \equiv \frac{v^2}{2} = \frac{\Lambda^2}{f_2}(-\lambda \pm \sqrt{\lambda^2 - f_2\mu^2/\Lambda^2}\,).$$

Because of $\mathcal{O}_1$, Higgs field must be rescaled. This will propagate to all Higgs couplings (with itself and all other particles). For example

$$\mathcal{L}_M = (M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu)\left((1-\frac{a_1}{2})\frac{2H}{v} + (1-a_1)\frac{H^2}{v^2}\right)$$

$$a_1 = \frac{f_1 v^2}{\Lambda^2}.$$

This can be directly measured in ZH production.

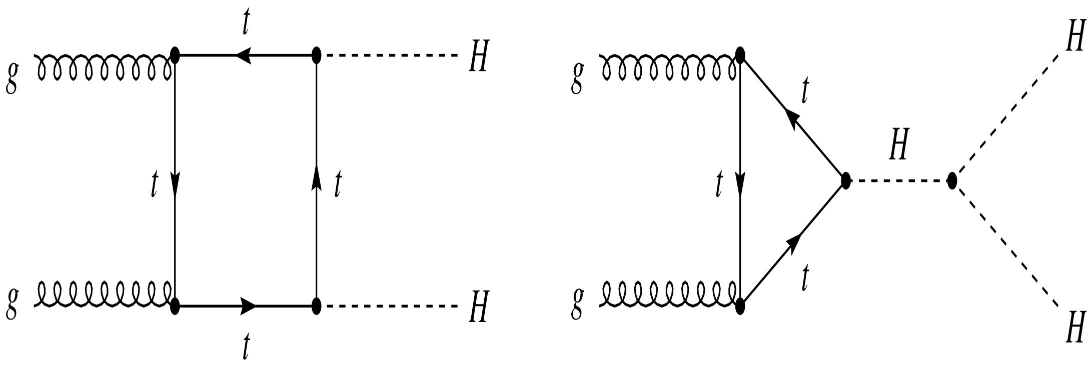
While for the self-couplings:

$$\mathcal{L}_{H^3} = -\frac{m_H^2}{2v}\left((1-\frac{a_1}{2}+\frac{2a_2}{3}\frac{v^2}{m_H^2})H^3 - \frac{2a_1 H\partial_\mu H\partial^\mu H}{m_H^2}\right)$$

$$\mathcal{L}_{H^4} = -\frac{m_H^2}{8v^2}\left((1-a_1+\frac{4a_2v^2}{m_H^2})H^4 - \frac{4a_1 H^2\partial_\mu H\partial^\mu H}{m_H^2}\right)$$

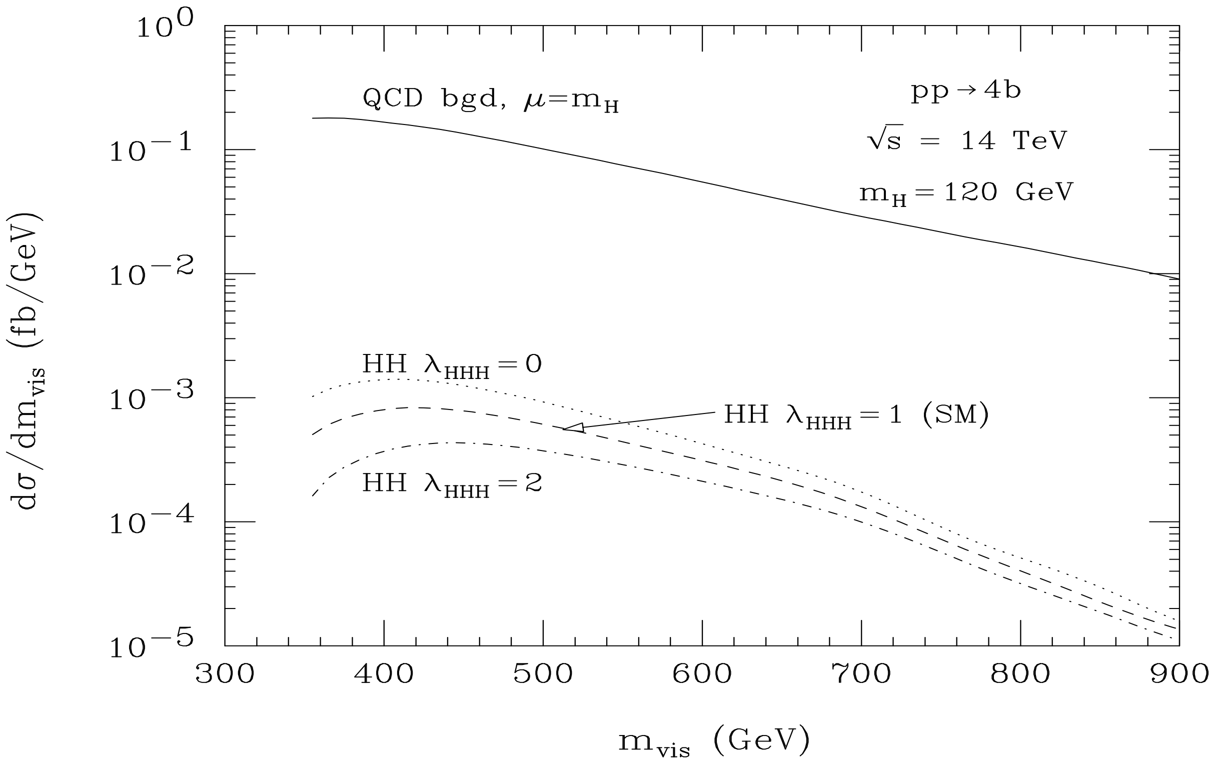
$$m_H^2 = 2\lambda v^2\left(1-a_1+\frac{a_2}{2\lambda}\right) \quad a_2 = \frac{f_2 v^2}{\Lambda^2}.$$

Triple Higgs vertex at the LHC



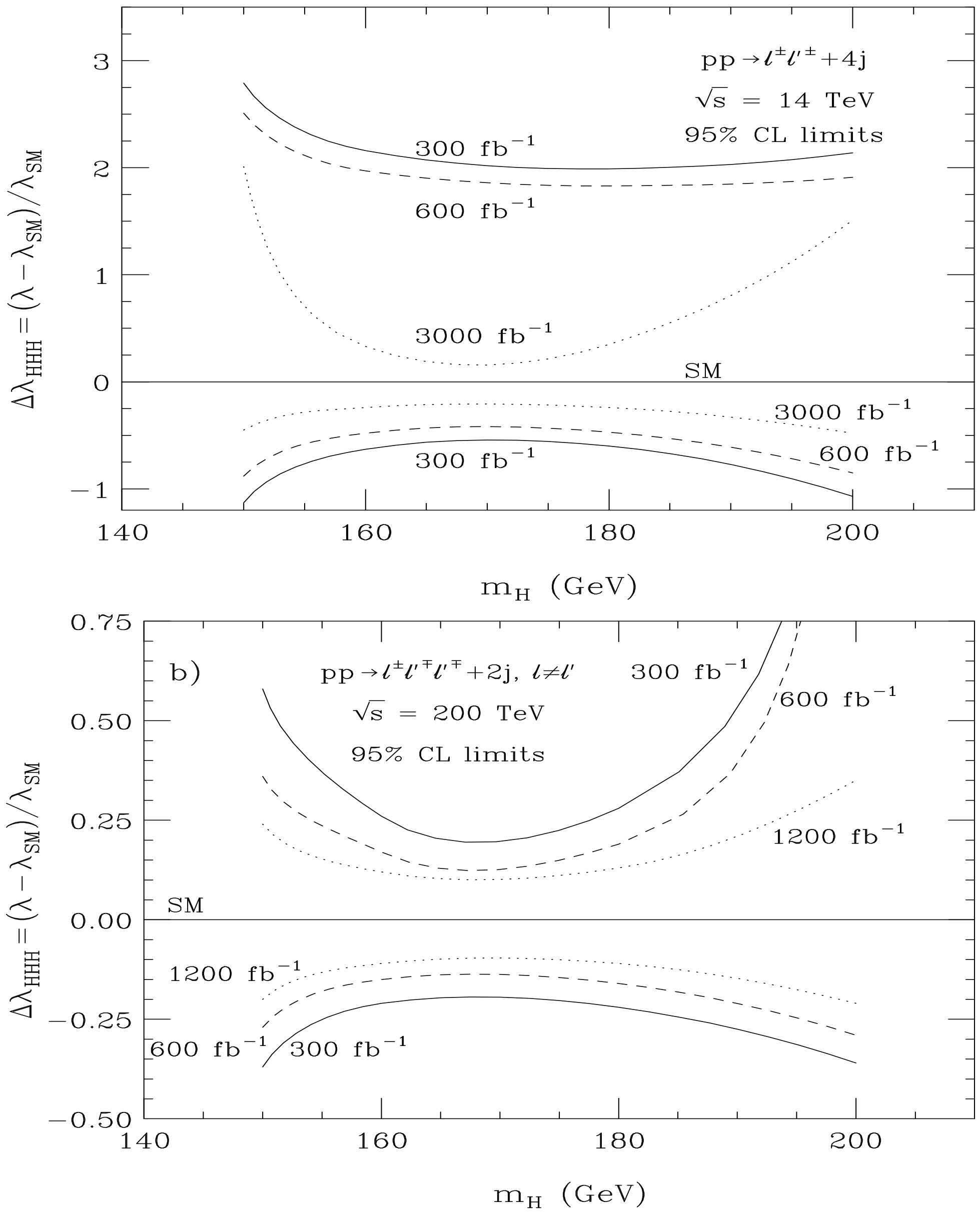
Main production mechanism at a hadronic machine

Looking for a the Higgs self-couplings of a light Higgs ($M_H < 2M_W$) is practically hopeless at a hadronic machine



Distribution of the visible invariant mass in $pp \to 4b$
from Baur, Plehn and Rainwater 2003.

Looking for a the Higgs self-couplings of a Higgs $(2M_W < M_H < 2M_Z)$

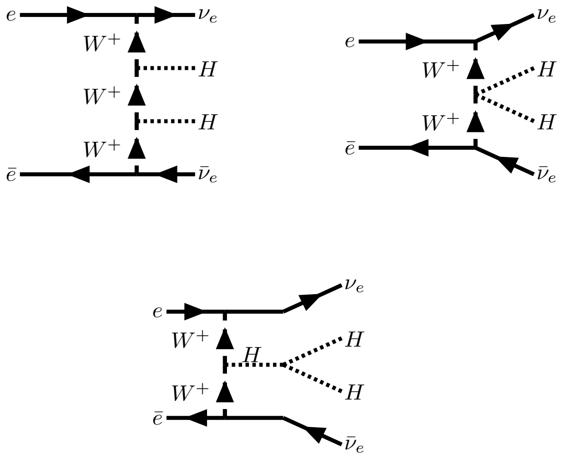


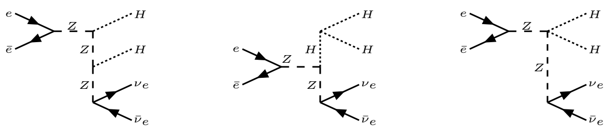
from Baur, Plehn and Rainwater 2002.

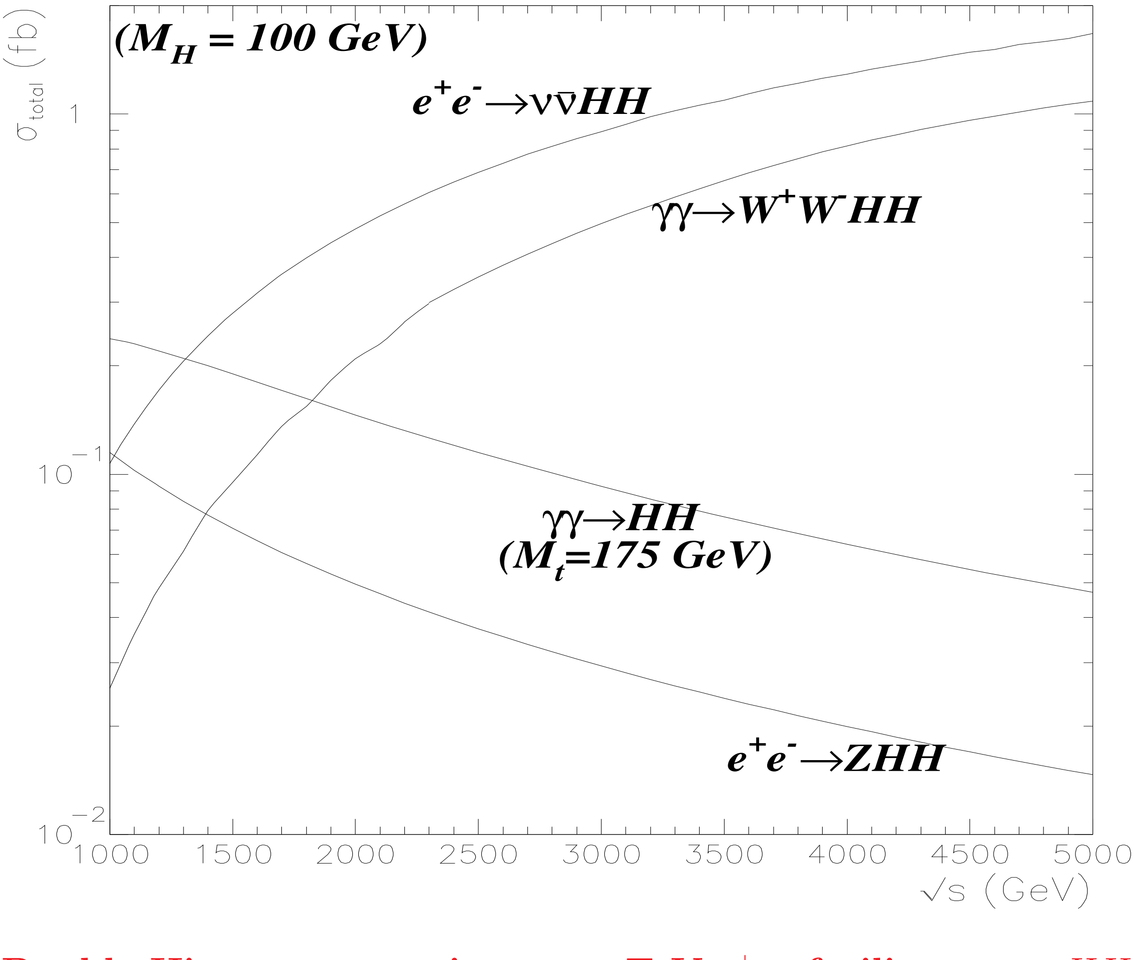
- At LHC (300fb^{-1}) precision of about $-60\% - 200\%$!
- At SLHC (3000fb^{-1}) precision of about $20\% - 30\%$, pile-up,...?
- VLHC, $8 - 25\%$ at 300fb^{-1} and $4 - 11\%$ 1200fb^{-1}

warning all these limits, on total rate....

Triple Higgs vertex at the Linear Colliders

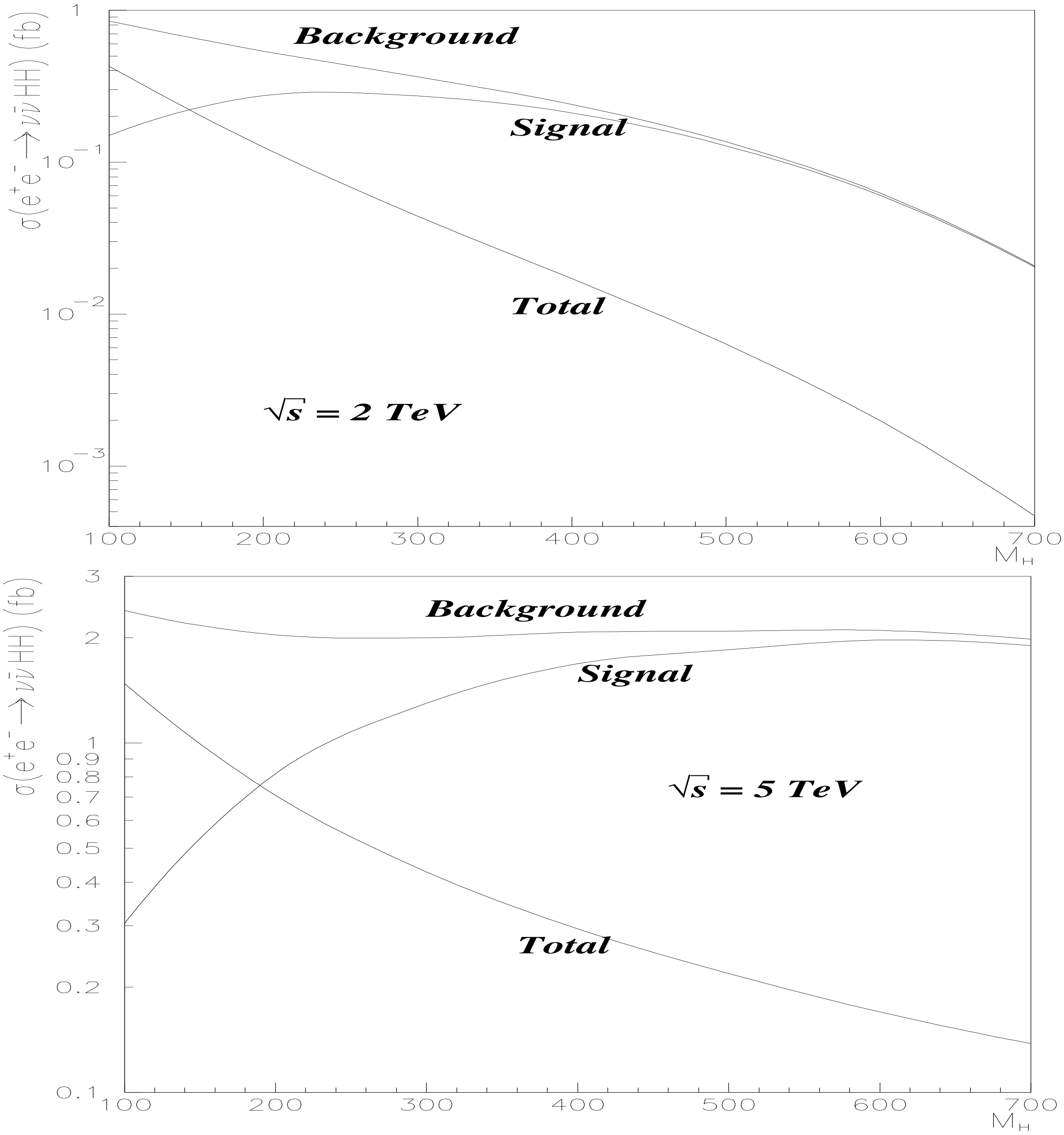






Double Higgs cross sections at a TeV e^+e^- facility. $\gamma\gamma \to HH$ from Jikia '94

F.B and E. Chopin 95



F.B and E. Chopin 95

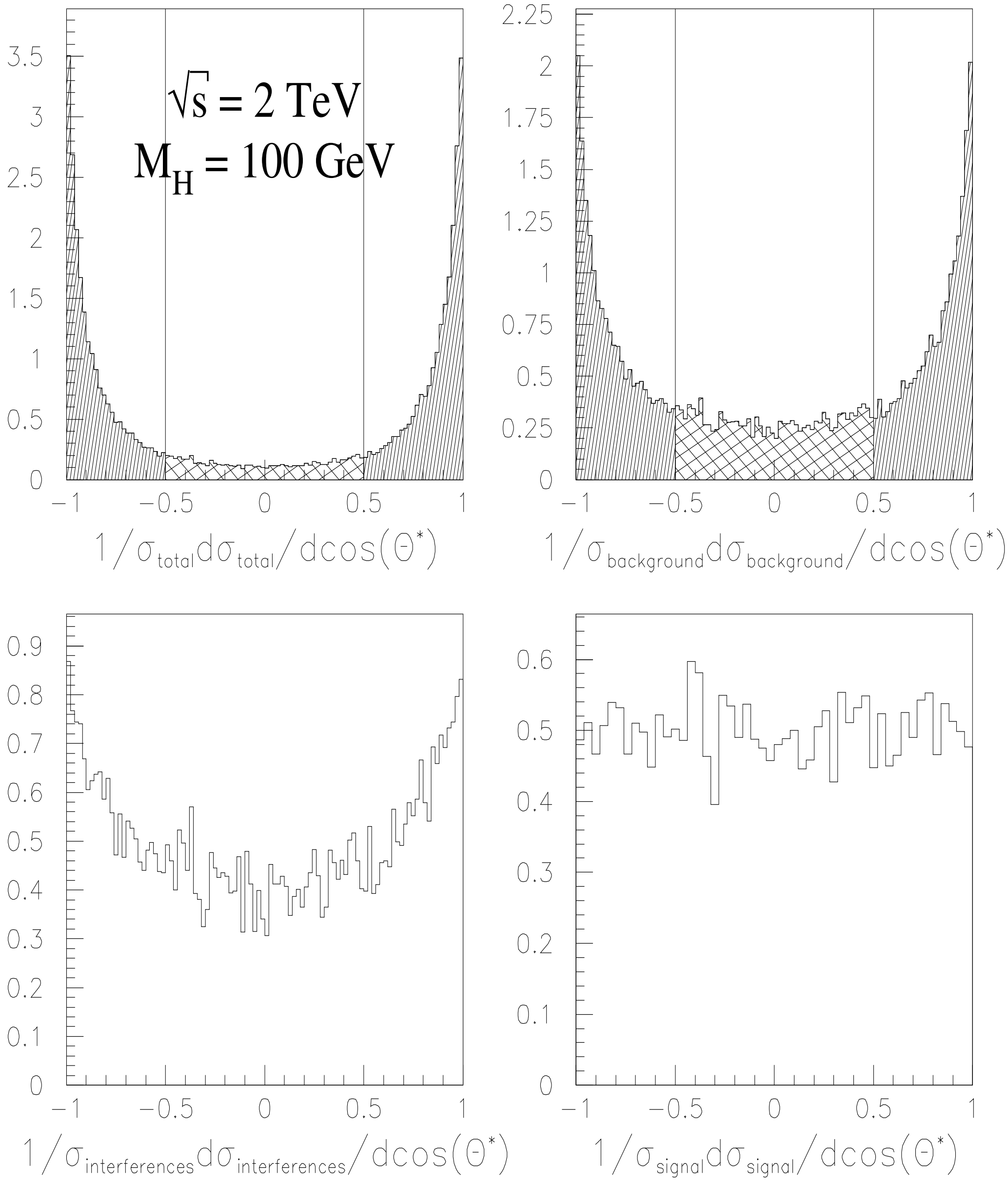
"Signal" is the contribution of the H^3 vertex. One sees that as M_H increases the cancellation between the H^3 contribution and the other contributions gets more and more delicate.

enhanced coupling effect

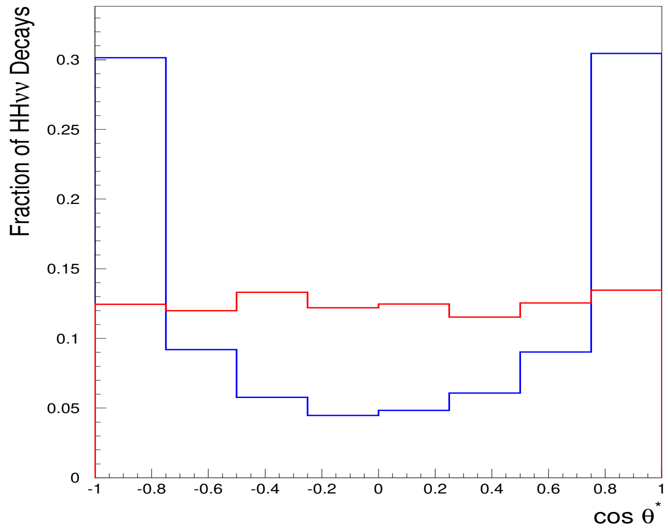
$$\tilde{\mathcal{M}}_{W_L^+W_L^- \to HH} \to \frac{g^2}{4} r(3h_3 - 2) + \ldots, \qquad r = M_H^2/M_W^2$$

Improving the limits and isolating the H^3 coupling

H^3 occurs in the subprocess $H^* \to HH$. This is like the decay of H^* which leads to an isotropic decay. Hence the angular distribution of the decay product in the rest frame of H^* should be flat.



FB+Chopin 95.



recent experimental simulation from Battaglia, Boos and Yao 2001

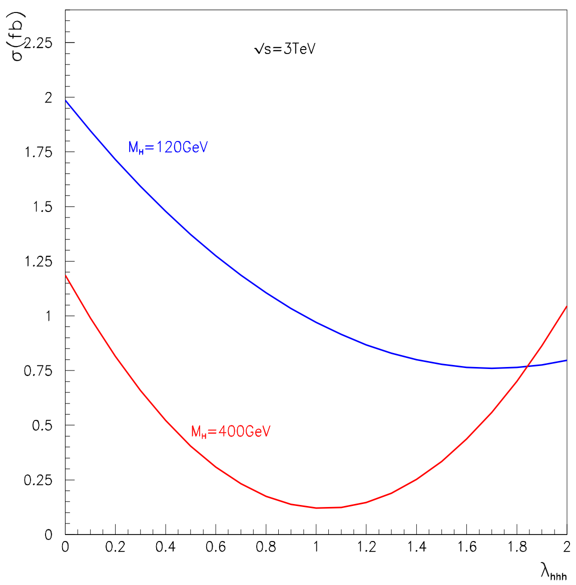
$\sqrt{s} = 3TeV, M_H = 120\mathrm{GeV}$

Summary of relative accuracies $\delta g_{HHH}/g_{HHH}$ for M_H = 120 GeV and $\int L$ = 1 ab^{-1} with the HHZ channel.

$\sqrt{s}$ (TeV)	σ_{HHZ} Only	M_{HH} Fit
0.5	$\pm$ 0.23 (stat)	$\pm$ 0.20 (stat)
0.8	$\pm$ 0.35 (stat)	$\pm$ 0.29 (stat)

Summary of relative accuracies $\delta g_{HHH}/g_{HHH}$ for $\int L$ = 5 ab^{-1} with the $HH\nu\bar{\nu}$ channel at 3 TeV (CLIC).

M_H (GeV)	$\sigma_{HH\nu\bar{\nu}}$ Only	$\lvert\cos\theta^*\rvert$ Fit
120	$\pm$ 0.094 (stat)	$\pm$ 0.070 (stat)
180	$\pm$ 0.140 (stat)	$\pm$ 0.080 (stat)



The case of SUSY and 2 Higgs doublet Model

based on F.B and A. Semenov '01.

For SUSY 2 doublets are needed.

$$H_1 = \begin{pmatrix} v_1 + \frac{1}{\sqrt{2}}(h_1 + i\varphi_1^0) \\ \varphi_1^- \end{pmatrix} \qquad H_2 = \begin{pmatrix} \varphi_2^+ \\ v_2 + \frac{1}{\sqrt{2}}(h_2 + i\varphi_2^0) \end{pmatrix}$$

$$\begin{aligned}
&V_{eff} = \\
&(m_1^2 + \mu^2)|H_1|^2 + (m_2^2 + \mu^2)|H_2|^2 - [m_{12}^2(\epsilon H_1 H_2) + h.c.] \\
&+\frac{1}{2}[\frac{1}{4}(g^2 + g'^2) + \lambda_1](|H_1|^2)^2 + \frac{1}{2}[\frac{1}{4}(g^2 + g'^2) + \lambda_2](|H_2|^2)^2 \\
&+[\frac{1}{4}(g^2 - g'^2) + \lambda_3]|H_1|^2|H_2|^2 + [-\frac{1}{2}g^2 + \lambda_4](\epsilon H_1 H_2)(\epsilon H_1^* H_2^*) \\
&+(\frac{\lambda_5}{2}(\epsilon H_1 H_2)^2 + [\lambda_6|H_1|^2 + \lambda_7|H_2|^2](\epsilon H_1 H_2) + h.c.)
\end{aligned}$$

ϵ is the antisymmetric matrix with $\epsilon_{12} = -1$. g and g' are the SU(2) and U(1) gauge couplings.

μ is a supersymmetry preserving mass term.

- The case with all λ_i being zero corresponds to the original MSSM potential at tree-level. Thus at tree-level in the MSSM, the quartic coupling are gauge couplings.

- Morever, exact supersymmetry imposes that $m_1^2 = m_2^2 = m_{12} = 0$ so that no electroweak symmetry breaking ensues. m_1, m_2 and m_{12} are thus essential for electroweak symmetry breaking and encode also supersymmetry breaking. These dimensionful quantities are soft SUSY breaking parameters.

- As for the λ's, practically all analyses of the Higgs phenomenology have only viewed them as "soft" terms originating from higher orders loop effects. As known these loop effects can be substantial as they are enhanced by large Yukawa couplings (corrections are quartic in the top mass) and have kept the MSSM alive.

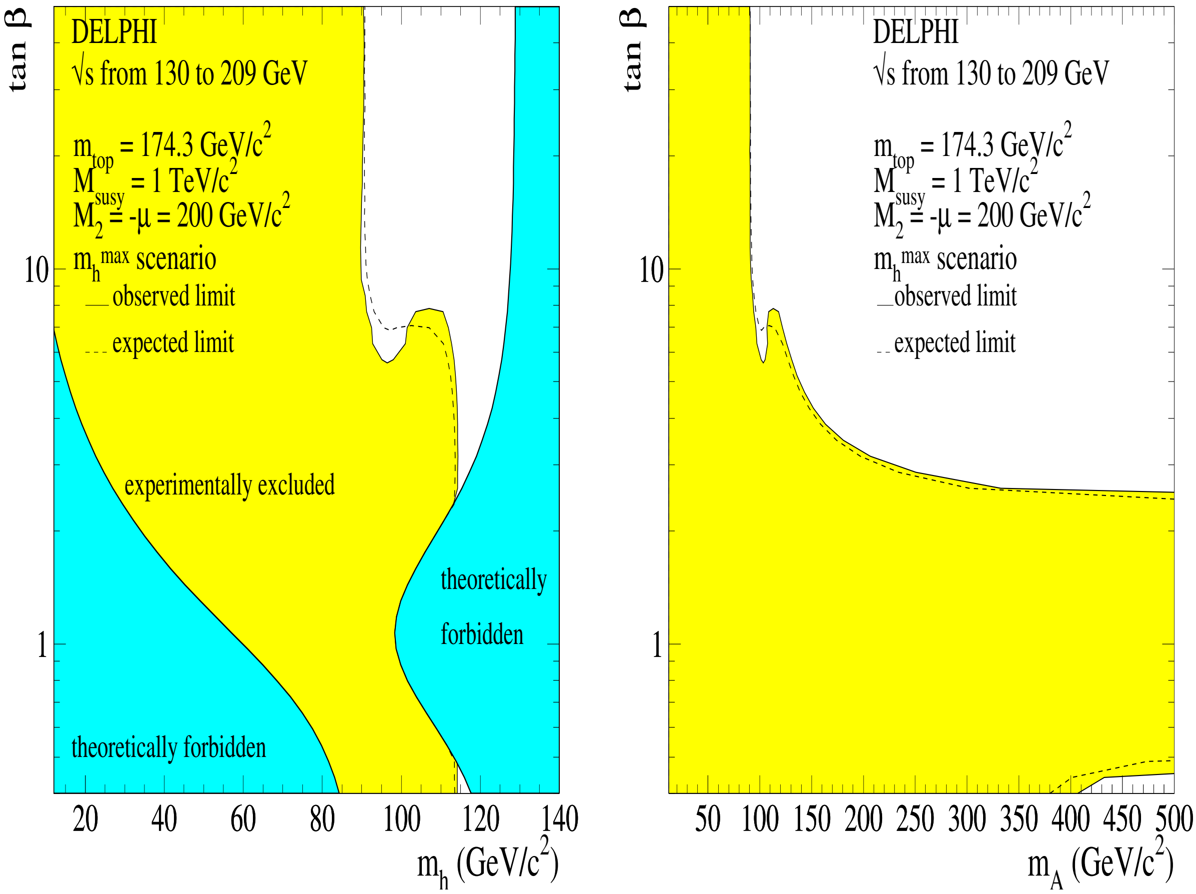
  In the MSSM at tree-level (all $\lambda_i = 0$)

$$\begin{aligned} M_{H,h}^2 &= \frac{1}{2}\left\{M_A^2 + M_Z^2 \pm \sqrt{(M_A^2 + M_Z^2)^2 - M_A^2 M_Z^2 c_{2\beta}^2}\right\} \\ &\hookrightarrow M_h \leq M_Z \\ c_{\beta-\alpha}^2 &= \frac{M_h^2(M_Z^2 - M_h^2)}{M_H^2 - m_h^2)(M_H^2 + M_h^2 - M_Z^2)}. \end{aligned} \tag{0.1}$$

  Including the leading 1-loop susy corrections

$$\begin{aligned} \lambda_2^{\tilde{t}_1,\tilde{t}_2} &= \frac{3}{32\pi^2}\frac{g^4 m_t^4}{s_\beta^4 M_W^4}\left\{\log(M_S^2/m_t^2) + \frac{A_t^2}{M_S^2}(1 - A_t^2/12M_S^4)\right\} \sim .2 \\ &\text{for} \quad A_t, M_S \sim 1TeV \end{aligned}$$

Present Limits on the Minimal MSSM Higgs



- However it may well be that models of supersymmetry breaking can provide a *direct* contribution to these parameters, for instance through non-renormalisable operators in the Kähler potential. Technically these contributions would be deemed *hard* . Indeed, it has been stressed recently (Polonski and SU, SP Martin) that if these parameters are related to the source of soft SUSY breaking then they would not destabalise the scalar potential and would evade the "un-natural" quadratic divergence problem, thus leading to a viable model. In such circumstances such terms may lead to **the lightest SUSY Higgs with a mass much in excess of 150GeV and with "no un-naturalessness" dilemma.** Therefore, the reconstruction of the Higgs potential is crucial. In a different context, supersymmetric models with a warped fifth dimension (Casas et al.,), it was shown that some of the quartic couplings (apart from the usual ones of gauge origin) can even be supersymmetric and originate from a non-minimal Kähler potential.

  Supersymmetry breaking terms also contribute to the latter as well as to the quadratic terms (and even tri-linear through the radion). However to obtain a satisfactory electroweak breaking in this scheme constrains the parameters such that $\tan\beta = 1$ is picked up (see below the definition of $\tan\beta$).

We will follow a more general context, assuming the quadratic terms ($m_{1,2,12}$) to fulfill the usual conditions for a stable minimum with non-vanishing vacuum expectation values. The minimisation of the potential, and the absence of tadpoles, imposes the following constraint on the the "soft" SUSY parameters m_1, m_2: a trade-off

$$
\begin{aligned}
m_1^2 &= -m_{12}^2 t_\beta - \mu^2 - M_Z^2 c_{2\beta}/2 + v^2(-\lambda_1 c_\beta^2 - \lambda_3 s_\beta^2 - \lambda_4 s_\beta^2 - \lambda_5 s_\beta^2 \\
&+ 3\lambda_6 c_\beta s_\beta + \lambda_7 s_\beta^3/c_\beta) \\
m_2^2 &= -m_{12}^2/t_\beta - \mu^2 + M_Z^2 c_{2\beta}/2 + v^2(-\lambda_2 s_\beta^2 - \lambda_3 c_\beta^2 - \lambda_4 c_\beta^2 - \lambda_5 c_\beta^2 \\
&+ \lambda_6 c_\beta^3/s_\beta + 3\lambda_7 c_\beta s_\beta)
\end{aligned}
$$

where $v^2 = v_1^2 + v_2^2 = 2M_W^2/g^2$, $t_\beta = tan\,\beta = v_2/v_1$, $s_\beta = \sin\beta$ and so on.

Then, the parameter m_{12}^2 can be fixed if we choose M_A, the pseudo-scalar Higgs mass, as an independent variable:

$$
m_{12}^2 = -c_\beta s_\beta(M_A^2 + v^2(2\lambda_5 - \lambda_6/t_\beta - \lambda_7 t_\beta)).
$$

Higgs masses and couplings to ordinary particles

At this stage, beside M_A and t_β, there are **seven** independent parameters. Luckily some of these enter the expressions of the Higgs masses and the couplings to fermions and vector bosons.

The mass of the charged Higgs boson reads as

$$M_{H^\pm}^2 = M_A^2 + M_W^2 - v^2(\lambda_4 - \lambda_5).$$

The two CP-even Higgs states are determined by the mixing angle α. h (H) will denote the lightest (heaviest) $\mathcal{CP}$ even Higgs while

$$\begin{aligned}
tan\,2\alpha &= \frac{M_A^2 s_{2\beta} + M_Z^2 s_{2\beta} - 2v^2(s_{2\beta}(\lambda_3+\lambda_4) - 2c_\beta^2\lambda_6 - 2s_\beta^2\lambda_7)}{M_A^2 c_{2\beta} - M_Z^2 c_{2\beta} - 2v^2(\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_5 c_{2\beta} - (\lambda_6-\lambda_7)s_{2\beta})} \\
M_H^2 &= M_Z^2 c_{\alpha+\beta}^2 + M_A^2 s_{\alpha-\beta}^2 \\
&\quad +2v^2(\lambda_1 c_\alpha^2 c_\beta^2 + \lambda_2 s_\alpha^2 s_\beta^2 + 2(\lambda_3+\lambda_4)c_\alpha c_\beta s_\alpha s_\beta + \lambda_5(c_\alpha^2 s_\beta^2 + s_\alpha^2 c_\beta^2) \\
&- 2s_{\alpha+\beta}(\lambda_6 c_\alpha c_\beta + \lambda_7 s_\alpha s_\beta)) \\
M_h^2 &= M_Z^2 s_{\alpha+\beta}^2 + M_A^2 c_{\alpha-\beta}^2 \\
&\quad +2v^2(\lambda_1 s_\alpha^2 c_\beta^2 + \lambda_2 c_\alpha^2 s_\beta^2 - 2(\lambda_3+\lambda_4)c_\alpha c_\beta s_\alpha s_\beta + \lambda_5(c_\alpha^2 c_\beta^2 + s_\alpha^2 s_\beta^2) \\
&+ 2c_{\alpha+\beta}(\lambda_6 s_\alpha c_\beta - \lambda_7 c_\alpha s_\beta))
\end{aligned}$$

It is important to stress that the bulk of the radiative corrections, especially those from the stop-top and sbottom-bottom sector can be effectively parametrised in terms of the λ_i, see for instance (Carena et al,). The decoupling limit is usually defined as $M_A \gg M_Z$, we will extend this to mean also $M_A \gg v$.

In the decoupling limit

$$M_A^2 c_{\beta-\alpha} s_{\beta-\alpha} \sim M_A^2 c_{\beta-\alpha} \rightarrow$$
$$s_{2\beta} c_{2\beta} \left\{ M_Z^2 - v^2 \left(\lambda_3 + \lambda_4 + \lambda_5 + (\lambda_6 - \lambda_7) t_{2\beta} - \lambda_1 \frac{c_\beta^2}{c_{2\beta}} + \lambda_2 \frac{s_\beta^2}{c_{2\beta}} - \frac{\lambda_6}{t_\beta} - \lambda_7 t_\beta \right) \right\}$$

These simple considerations already show that if some of the λ_i are not too tiny one should observe their effects by measuring the Higgs masses and also the Higgs couplings to ordinary fermions. This generalisation as pointed out by Polonsky and Su for example can allow for a lightest Higgs mass in excess of 150GeV, say,independently of t_β.

We will always express the dependence in the mixing angle α through $c_{\beta-\alpha}$ and $s_{\beta-\alpha}$ since these enter the couplings of h (and H) to vector bosons and to fermions. In units of the $\mathcal{SM}$ Higgs couplings, the couplings to vector bosons are

$g_{hVV,HVV} = s_{\beta-\alpha}, c_{\beta-\alpha}$, while for instance

$g_{hb\bar{b}} = s_{\beta-\alpha} - t_\beta c_{\beta-\alpha}$.

Therefore these combinations, especially if t_β has been extracted from other SUSY processes, could be extracted once the light Higgs has been produced at the linear collider, and after allowing for some (important) QCD and in some cases model dependent vertex corrections.

It is important to keep in mind for later reference that in the decoupling limit we have:

$$c_{\beta-\alpha}\ ,\ (M_A^2 - M_H^2) \rightarrow \mathcal{O}(1/M_A^2)$$

Decoupling with

$$r = \frac{2M_Z^2}{M_A^2}$$

X	$g_{X\bar{u}u}$	$g_{X\bar{d}d}$	g_{XVV}
h	$\cos\alpha/\sin\beta$ $\rightarrow s_{\beta-\alpha} + c_{\beta-\alpha}/t_\beta$ $\rightarrow 1 - r/t_\beta$	$-\sin\alpha/\cos\beta$ $s_{\beta-\alpha} - c_{\beta-\alpha}t_\beta$ $1 + rt_\beta$	$\sin(\beta-\alpha)$ $s_{\beta-\alpha}$ $1 - r^2/(2t_\beta^2)$
H	$\sin\alpha/\sin\beta$ $\rightarrow\ 1/t_\beta$	$\cos\alpha/\cos\beta$ $\rightarrow\ t_\beta$	$\cos(\beta-\alpha)$ $r\sin(4\beta)/4 \rightarrow r/t_\beta$
A	$1/t_\beta$	t_β	0

Therefore precise measurements of these couplings, together with some of the Higgs masses, are already a window on the Higgs potential

The Higgs tri-linear self-couplings

What we usually see for the Higgs self-couplings is not very telling at all. Moreover we get the impression that since we have so many tri-linear couplings we could, in principle, reconstruct the remaining 3 parameters after having measured the masses:

in units of

$$g_h = \frac{2M_W}{g} = \sqrt{2}v$$

we have

$$\begin{aligned}
g_{hhh} &= 3g_h\lambda_{hhh} \\
\lambda_{hhh} &= -\frac{e^2}{s_{2w}^2} s_{\beta+\alpha} c_{2\alpha} \\
&+ \lambda_1\, c_\beta s_\alpha^3 - \lambda_2\, c_\alpha^3 s_\beta + \frac{1}{2}(\lambda_3+\lambda_4+\lambda_5)\, s_{2\alpha} c_{\beta+\alpha} \\
&+ \lambda_6 s_\alpha^2 (c_{\beta+\alpha} + 2c_\alpha c_\beta) + \lambda_7 c_\alpha^2 (c_{\beta+\alpha} - 2s_\alpha s_\beta)
\end{aligned}$$

$$\begin{aligned}
g_{Hhh} &= -3g_h\lambda_{Hhh} \\
\lambda_{Hhh} &= \frac{e^2}{s_{2w}^2}\left(s_{\beta+\alpha}s_{2\alpha} - \frac{1}{3}c_{\beta-\alpha}\right) \\
&+ \lambda_1\, c_\beta s_\alpha^2 c_\alpha + \lambda_2\, s_\beta c_\alpha^2 s_\alpha - \frac{1}{2}(\lambda_3+\lambda_4+\lambda_5)\,(s_{2\alpha}s_{\beta+\alpha} - \frac{2}{3}c_{\beta-\alpha}) \\
&+ \lambda_6 s_\alpha (c_\beta c_{2\alpha} + c_\alpha c_{\beta+\alpha}) - \lambda_7 c_\alpha (s_\beta c_{2\alpha} + s_\alpha c_{\beta+\alpha})
\end{aligned}$$

$$\begin{aligned}
g_{hHH} &= 3g_h\lambda_{hHH} \\
\lambda_{hHH} &= \frac{e^2}{s_{2w}^2}(c_{2\alpha}s_{\beta+\alpha} - \frac{2}{3}s_{\beta-\alpha}) \\
&+ c_\alpha s_\alpha(\lambda_1 c_\alpha c_\beta - \lambda_2\, s_\alpha s_\beta) - \frac{(\lambda_3+\lambda_4+\lambda_5)}{2}\,(s_{\beta+\alpha}c_{2\alpha} - \frac{s_{\beta-\alpha}}{3}) \\
&+ (\lambda_6 c_\alpha (c_\beta c_{2\alpha} - s_\alpha s_{\beta+\alpha}) + \lambda_7 s_\alpha (c_{2\alpha} s_\beta + c_\alpha s_{\beta+\alpha})
\end{aligned}$$

$$
\begin{aligned}
g_{hAA} &= g_h \lambda_{hAA} \\
\lambda_{hAA} &= -\frac{e^2}{s_{2w}^2} s_{\beta+\alpha} c_{2\beta} \\
&+ \lambda_1 \, c_\beta s_\beta^2 s_\alpha - \lambda_2 \, s_\beta c_\beta^2 c_\alpha - (\lambda_3 + \lambda_4 + \lambda_5) \, (s_\beta^3 c_\alpha - s_\alpha c_\beta^3) + 2\lambda_5 s_{\beta-\alpha} \\
&+ \lambda_6 s_\beta (c_{2\beta} s_\alpha + c_\beta s_{\beta+\alpha}) + \lambda_7 c_\beta (c_{2\beta} c_\alpha - s_\beta s_{\beta+\alpha})
\end{aligned}
$$

$$
\begin{aligned}
g_{hH^+H^-} &= g_h \lambda_{hH^+H^-} \\
\lambda_{hH^+H^-} &= \lambda_{hAA} - \frac{e^2}{2 s_w^2} s_{\beta-\alpha} + (\lambda_4 - \lambda_5) s_{\beta-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
g_{HHH} &= -3 g_h \lambda_{HHH} \\
\lambda_{HHH} &= \frac{e^2}{s_{2w}^2} c_{2\alpha} c_{\beta+\alpha} \\
&+ c_\alpha^3 c_\beta \lambda_1 + s_\alpha^3 s_\beta \lambda_2 + \frac{(\lambda_3 + \lambda_4 + \lambda_5)}{2} \, s_{2\alpha} s_{\beta+\alpha} \\
&- \lambda_6 c_\alpha^2 (s_\beta c_\alpha + 3 s_\alpha c_\beta) - \lambda_7 s_\alpha^2 (c_\beta s_\alpha + 3 c_\alpha s_\beta)
\end{aligned}
$$

$$
\begin{aligned}
g_{HAA} &= -g_h \lambda_{HAA} \\
\lambda_{HAA} &= -\frac{e^2}{s_{2w}^2} c_{\beta+\alpha} c_{2\beta} \\
&+ \lambda_1 \, c_\beta s_\beta^2 c_\alpha + \lambda_2 \, s_\beta c_\beta^2 s_\alpha + (\lambda_3 + \lambda_4 + \lambda_5) \, (s_\beta^3 s_\alpha + c_\beta^3 c_\alpha) - 2\lambda_5 c_{\beta-\alpha} \\
&+ \lambda_6 s_\beta (c_\beta c_{\beta+\alpha} + c_\alpha c_{2\beta}) - \lambda_7 c_\beta (s_\beta c_{\beta+\alpha} + s_\alpha c_{2\beta})
\end{aligned}
$$

$$
\begin{aligned}
g_{HH^+H^-} &= -g_h \lambda_{HH^+H^-} \\
\lambda_{HH^+H^-} &= \lambda_{HAA} + \frac{e^2}{2 s_w^2} c_{\beta-\alpha} + (\lambda_5 - \lambda_4) c_{\beta-\alpha}
\end{aligned}
$$

Note that the heavy Higgs masses, $M_H, M_A, M_{H^\pm}$, enter the formulae for the couplings only through combinations involving mass differences or $M_A^2 c_{\beta-\alpha}$ so that in effect the self-couplings do not grow with the heavy Higgs mass as occurs in the $\mathcal{SM}$. However written this way the expressions are not very telling, moreover

it is clear that some of the couplings must be related since after having measured the masses, there remains only three independent λ while there are 8 Higgs tri-linear couplings.

Expressing the self-couplings in the mass basis

The Higgs masses already constrain the Higgs potential, even if partially. Since the Higgs masses (or at least the lightest Higgs) will be measured first, the tri-linear being accessed only trough double Higgs production whose cross section is small, one should ask how much more do we learn from the measurement of the tri-linear couplings once we have measured the masses. For example, it is quite likely that the heavy Higgses are too heavy so that one only has access to the lightest $\mathcal{CP}$ even Higgs. In this situation one would have an extremely precise determination of its mass (either at the LHC or linear collider) and also a very precise determination of its couplings to fermions and (gauge bosons) and hence of the angle α (or the combination $\alpha - \beta$) at a linear collider. Apart from M_A, one can trade M_h, M_H, $M_{H^\pm}$, α for, for example, λ_1, λ_2, λ_3, λ_4 through

$$\begin{aligned}
\lambda_4 &= (M_A^2 + M_W^2 - M_{H\pm}^2)/v^2 + \lambda_5 \\
\lambda_1 &= \frac{M_h^2 s_\alpha^2 + M_H^2 c_\alpha^2 - M_Z^2 c_\beta^2 - M_A^2 s_\beta^2}{2v_1^2} - \lambda_5 t_\beta^2 + 2\lambda_6 t_\beta \\
\lambda_2 &= \frac{M_h^2 c_\alpha^2 + M_H^2 s_\alpha^2 - M_Z^2 s_\beta^2 - M_A^2 c_\beta^2}{2v_2^2} - \lambda_5/t_\beta^2 + 2\lambda_7/t_\beta \\
\lambda_3 &= \frac{(M_H^2 - M_h^2)c_\alpha s_\alpha + (M_A^2 + M_Z^2)c_\beta s_\beta}{2v_1 v_2} - \lambda_4 + \lambda_6/t_\beta + \lambda_7 t_\beta.
\end{aligned}$$

With this choice the different Higgs self couplings are given in terms of the Higgs masses and the remaining $\lambda_{5,6,7}$. This choice of basis seems to be the most natural, since the structures that affect λ_{1-4} appear already at "tree-level" and thus should be the most affected by radiative corrections. This is so in the MSSM where the largest corrections at moderate t_β is in λ_2. , define first

$$\begin{aligned}\lambda_a &= \lambda_5 s_{2\beta} - \frac{\lambda_6+\lambda_7}{2} - \frac{\lambda_6-\lambda_7}{2} c_{2\beta} \\ \lambda_b &= \lambda_5 c_{2\beta} + \frac{\lambda_6-\lambda_7}{2} s_{2\beta}\end{aligned}$$

$$\begin{aligned}\lambda_{hhh} &= -\frac{1}{2v^2} s_{\beta-\alpha}\, m_h^2 \\ &+ \frac{c^2_{\beta-\alpha}}{s_\beta c_\beta}\left(s_{\beta-\alpha}\lambda_a + c_{\beta-\alpha}\lambda_b + \frac{(M_A^2 - m_h^2)}{2v^2}(s_{2\beta}s_{\beta-\alpha} + c_{2\beta}c_{\beta-\alpha})\right)\end{aligned}$$

This shows that in the decoupling limit the hhh coupling is completely determined from the measurement of the Higgs mass, m_h and the coupling to WWh/ZZh, $s_{\beta-\alpha}$, both of which should be determined quite precisely from $e^+e^- \rightarrow Zh$ at the LC. The λ_a and more so λ_b are totally screened. The result in the decoupling limit is no surprise as it reproduces exactly the $\mathcal{SM}$ result. In this regime one essentially have only one Higgs doublet and if one restricts oneself to dim-4 operators, one reproduces the $\mathcal{SM}$ exactly.

$$\begin{aligned}\lambda_{Hhh} &= c_{\beta-\alpha}\left\{ \frac{4\lambda_a}{3s_{2\beta}}(1 - \frac{3}{2}c^2_{\beta-\alpha}) + \frac{2\lambda_b}{s_{2\beta}} c_{\beta-\alpha}s_{\beta-\alpha} \;+\; \frac{1}{6v^2}\left((3M_H^2 - 2M_h^2)\right.\right. \\ &+ \left.\left. 4(M_A^2 - M_H^2) + \frac{c_{\beta-\alpha}\big(s_{2\beta}c_{\beta-\alpha} - c_{2\beta}s_{\beta-\alpha}\big)}{s_\beta c_\beta}(2M_h^2 + M_H^2 - 3M_A^2)\right)\right\}\end{aligned}$$

In the decoupling limit

$$\lambda_{Hhh} = \frac{c_{\beta-\alpha}}{2v^2} M_H^2$$

Since this coupling is most easily extracted from $H \rightarrow hh$, M_H would have (together with $c_{\beta-\alpha}$) already been measured, thus the new physics effects are again screened.

The remaining couplings involve at least two heavy Higgses an thus would be more difficult to measure. Nonetheless let us express them in terms of masses. All the couplings will be written as

$$g_{H_iH_jH_k} = \frac{g_h}{s_{2\beta}}\lambda'_{H_iH_jH_k}$$

$$\lambda'_{hAA} = 2s_{\beta-\alpha}\lambda_a + 2c_{\beta-\alpha}\lambda_b \; + \; \frac{1}{2v^2}\left(c_{\beta-\alpha}c_{2\beta}(2M_A^2 - 2M_h^2) - M_h^2 s_{2\beta}s_{\beta-\alpha}\right)$$

$$\begin{aligned}\lambda'_{hHH} \;&=\; s_{\beta-\alpha}\left\{\frac{1}{2v^2}\left[\left(M_h^2 + 2(M_H^2 - M_A^2)\right)s_{2\alpha} - 2M_A^2 c_{\beta-\alpha}s_{\beta+\alpha}\right]\right.\\ &+\; \left. 2(\lambda_a(1-3c^2_{\beta-\alpha}) + 3c_{\beta-\alpha}s_{\beta-\alpha}\lambda_b)\right\}\end{aligned}$$

$$\begin{aligned}\lambda'_{hH^+H^-} \;&=\; -2\left\{\frac{1}{2v^2}\left[(c_\alpha c_\beta^3 - s_\alpha s_\beta^3)M_h^2 - (M_A^2 - M_{H^\pm}^2)c_{\beta+\alpha} - M_{H^\pm}^2 c_{2\beta}c_{\beta-\right.\right.\\ &\left. -(s_{\beta-\alpha}\lambda_a + c_{\beta-\alpha}\lambda_b)\right\}\end{aligned}$$

$$\begin{aligned}\lambda'_{HHH} \;&=\; -6\left\{\frac{1}{2v^2}\left[s_{\beta+\alpha}\left((M_H^2 - M_A^2) + c^2_{\beta-\alpha}M_A^2\right) - s_\alpha c_\alpha c_{\beta-\alpha}M_H^2\right]\right.\\ &\left. +s^2_{\beta-\alpha}(s_{\beta-\alpha}\lambda_b - c_{\beta-\alpha}\lambda_a)\right\}\end{aligned}$$

$$\begin{aligned}\lambda'_{HAA} \;&=\; -2\left\{\frac{1}{2v^2}\left[(s_\alpha c_\beta^3 + c_\alpha s_\beta^3)(M_H^2 - M_A^2) - c_{\beta-\alpha}c_\beta s_\beta M_A^2\right]\right.\\ &\left. +s_{\beta-\alpha}\lambda_b - c_{\beta-\alpha}\lambda_a\right\}\end{aligned}$$

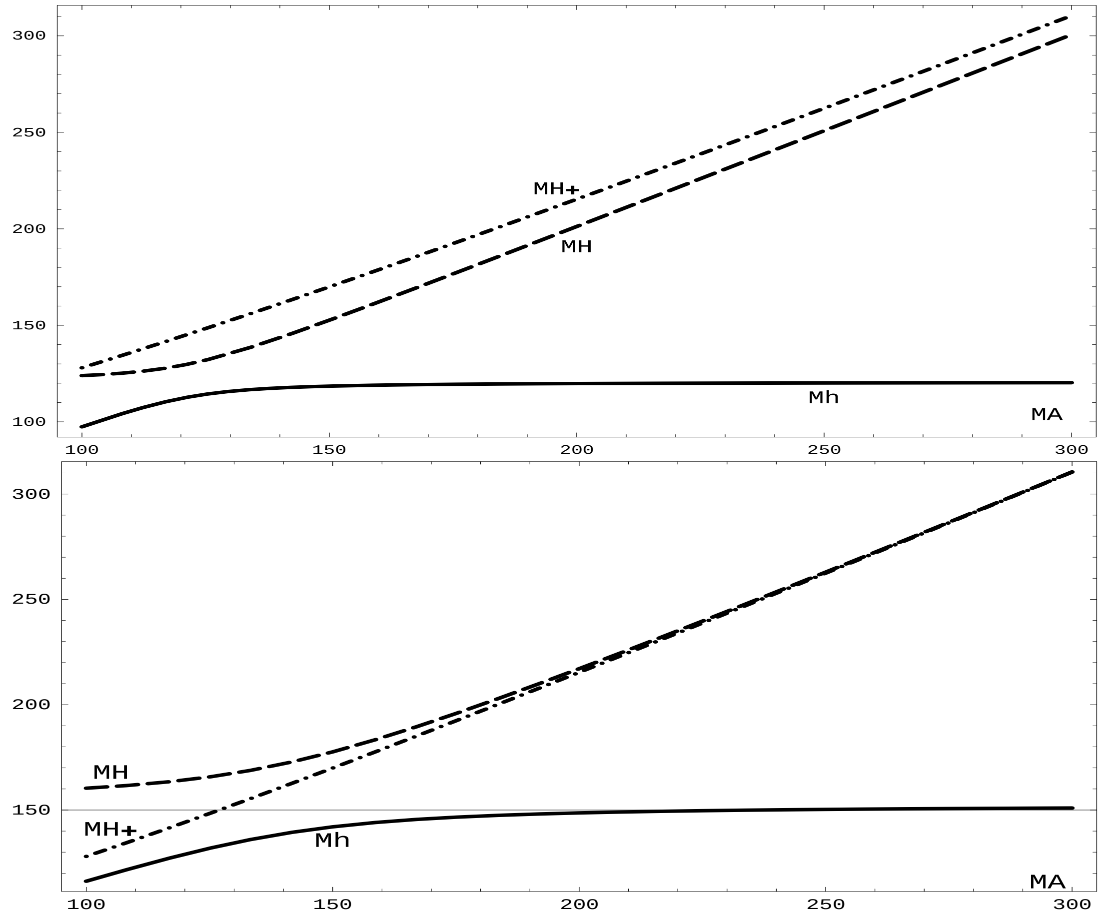
$$\begin{aligned}\lambda'_{HH^+H^-} \;&=\; -2\left\{\frac{1}{2v^2}\left[s_{\beta+\alpha}(M_H^2 - M_A^2) + c_{\beta-\alpha}c_\beta s_\beta(2M_{H^\pm}^2 - M_H^2)\right]\right.\\ &\left. +s_{\beta-\alpha}\lambda_b - c_{\beta-\alpha}\lambda_a\right\}\end{aligned}$$

SUMMARY
on tri-linear couplings from leading operators

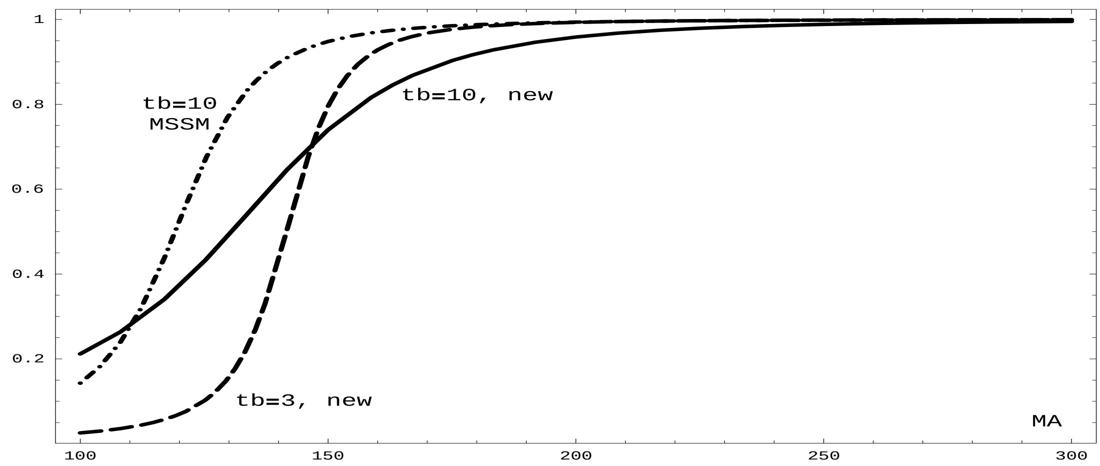
In the physical basis and especially by explicitly displaying the dependence in the mixing angle α through the combination $c_{\beta-\alpha}, s_{\beta-\alpha}$ shows that for couplings involving at least two heavy Higgses

- The dependence in the parameters λ_i only appear through the combination λ_a *or* λ_b. Thus one combination is not accessed in the tri-linear couplings.

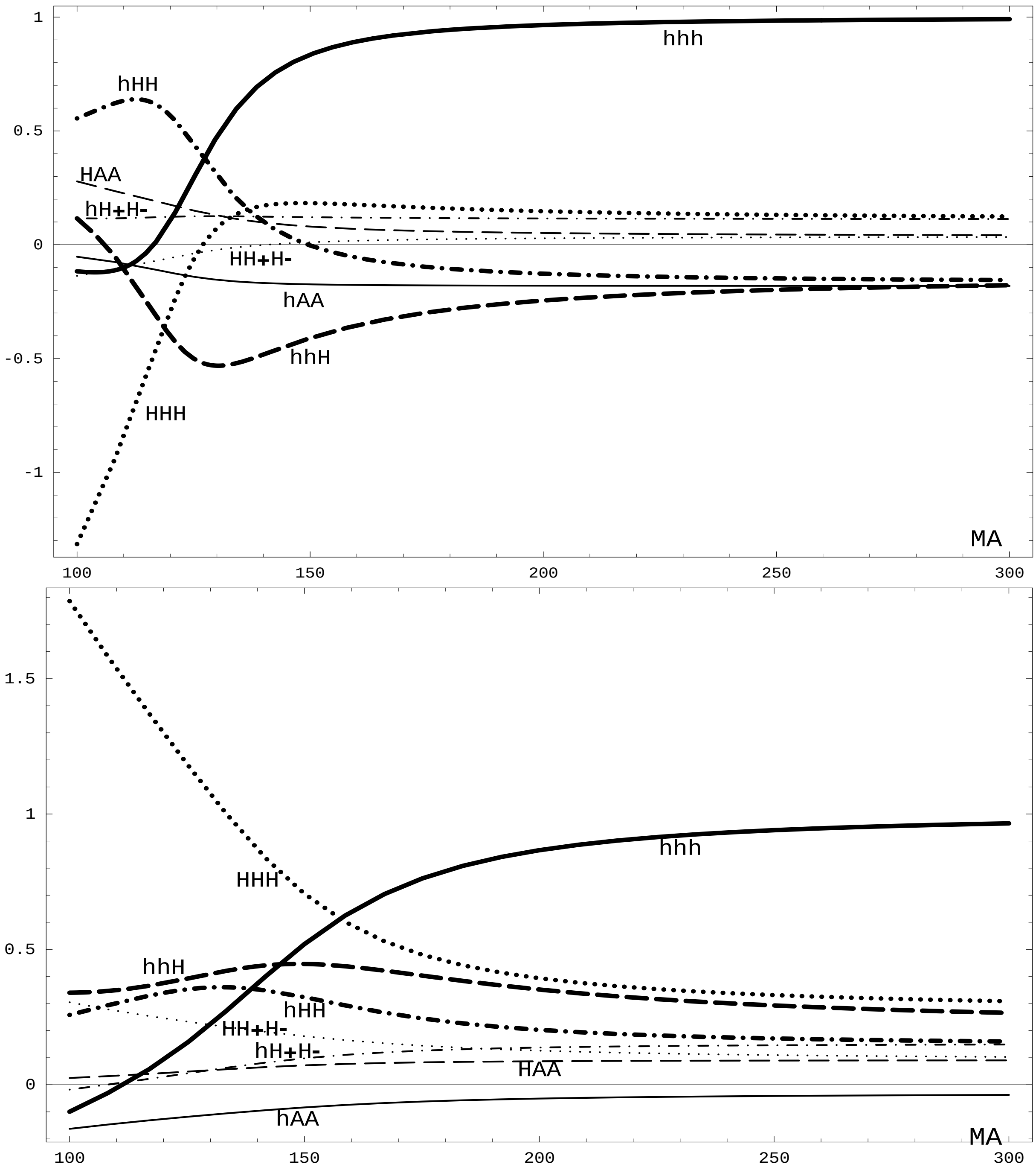
- In the decoupling limit, $m_h \ll M_A, M_H, M_A$, the self-couplings with two heavy Higgses, $(hAA, hHH$ and $hH^+H^-)$, only λ_a is accessible, while the self-couplings with heavy Higgses only, (HHH, HAA, HH^+H^-) only λ_b may be accessed.



Higgs mass spectrum without "hard" terms but with $A_t = 1000\text{GeV}$, $M_S = 800\text{GeV}$, $\mu = -300\text{GeV}$ and $\tan\beta = 10$ and with the inclusion of additional terms with $\lambda^{\rm new}_{1-5} = -\lambda^{\rm new}_{6,7} = .1$.



$s^2_{\beta-\alpha}$ as a function of M_A. MSSM refers to $\lambda_i = 0$ with $A_t = 1000\text{GeV}$, $M_S = 800\text{GeV}$,$\mu = -300\text{GeV}$ while "new" has $\lambda^{\rm new}_{1-5} = -\lambda^{\rm new}_{6,7} = .1$.



$g_{H^iH^jH^k}/\tilde{g}^{\rm SM}_{hhh}$ with $\tilde{g}^{\rm SM}_{hhh} = -3M_h^2/2v^2$. The SM Higgs mass is identified with M_h (and thus varies with M_A). The curve in the first panel is the usual MSSM.

Quartic Higgs couplings

To access the remaining combination we need to consider the quartic couplings. We will write the quartic couplings directly in the physical basis.

Defining

$$\lambda_c = 2c_{2\beta}\lambda_b - \lambda_5 = \lambda_5 c_{4\beta} + \frac{\lambda_6 - \lambda_7}{2} s_{4\beta}$$

$$\begin{aligned}
\lambda_{hhhh} &= -\frac{3M_h^2}{2v^2}(1 - c_{\beta-\alpha}^2) \\
&- \frac{3}{2v^2 s_\beta^2 c_\beta^2} c_{\beta-\alpha}^2 \left\{ s_{2\beta} s_{\beta-\alpha} M_h^2 + s_\alpha^2 c_\alpha^2 M_H^2 + c_{\beta-\alpha}^2 c_{\beta+\alpha}^2 (M_h^2 - M_A^2) \right\} \\
&+ \frac{3}{2v^2 s_\beta^2 c_\beta^2} c_{\beta-\alpha}^2 \left\{ s_{2\beta}\lambda_a + 2 s_{2\beta} c_{\beta-\alpha} s_{\beta-\alpha}\lambda_b + c_{\beta-\alpha}^2 \lambda_c \right\}
\end{aligned}$$

We see again that in the decoupling region one is not sensitive to any of the extra couplings, as expected since we recover the $\mathcal{SM}$ result with only the dim-4 operator.

$$\lambda_{hhhH} \;\; \rightarrow \;\; c_{\beta-\alpha} \left\{ s_{\beta-\alpha} s_{2\beta}\lambda_a + + c_{\beta-\alpha}(3 s_{2\beta}\lambda_b - 2 c_{\beta-\alpha} s_{\beta-\alpha}\lambda_c - 4 c_{\beta-\alpha}^2 s_{2\beta}\lambda_b) \right\}$$

Again, all the anomalous couplings are screened.

$$\lambda_{hhHH} \;\; \rightarrow \;\; 2 s_{2\beta}\lambda_a + 3 c_{\beta-\alpha}\left(s_{2\beta} s_{\beta-\alpha}\lambda_b + c_{\beta-\alpha} s_{\beta-\alpha}^2 \lambda_c - 2 c_{\beta-\alpha}^2 s_{\beta-\alpha} s_{2\beta}\lambda_b \right)$$

In $hhHH$ only λ_a may be accessible.

$$\lambda_{hHHH} \;\; \rightarrow \;\; s_{\beta-\alpha} s_{2\beta}\lambda_b + c_{\beta-\alpha}\left(s_{2\beta}\lambda_a + 2 s_{\beta-\alpha}^2 \lambda_c + 4 c_{\beta-\alpha} s_{\beta-\alpha} s_{2\beta}\lambda_b \right)$$

In $hHHH$ only λ_b may be accessible.

$$\lambda_{HHHH} \;\; \rightarrow \;\; s_{\beta-\alpha}^2 \left(s_{\beta-\alpha}^2 \lambda_c + s_{2\beta}\lambda_a - 2 c_{\beta-\alpha} s_{\beta-\alpha} s_{2\beta}\lambda_b \right)$$

In $HHHH$ both λ_a and λ_c may be accessible but not λ_b.

Using another parametrisation The screening property is a general result which does not depend on which independent parameters we keep beside the physical masses. Had we used another set of independent parameters beside the masses, the same phenomenon would have occurred and only two independent combinations out of the three parameters would enter the expression of the tri-linear couplings. Indeed with $\lambda_3, \lambda_5, \lambda_6$ as extra parameters, the role of $\lambda_{a,b,c}$ is played by $\lambda'_{a,b,c}$, such that

$$\begin{aligned}
\lambda_a \to \lambda'_a &= -s_\beta c_\beta(\lambda_3 - \lambda_5) \\
\lambda_b \to \lambda'_b &= -\left(\frac{\lambda_3 + \lambda_5}{2} + c_{2\beta}\frac{\lambda_3 - \lambda_5}{2}\right) + \frac{c_\beta}{s_\beta}\lambda_6 \\
\lambda_c \to \lambda'_c &= 2\lambda'_b c_{2\beta} - \lambda_5
\end{aligned}$$

For instance,

$$\begin{aligned}
\lambda_{hhh} &= -\frac{1}{2v^2} s_{\beta-\alpha}\, m_h^2 \\
&- c^2_{\beta-\alpha}\frac{c_\alpha}{s_\beta}\frac{e^2}{s^2_{2W}} - c^2_{\beta-\alpha}\left(s_{\beta-\alpha}\lambda'_a + \frac{c_{\beta-\alpha}}{2c_\beta s^2_\beta}\lambda'_b\right) \\
&- \frac{c^2_{\beta-\alpha}}{2c_\beta s^2_\beta}\left\{M_h^2\left(c_\beta c_{2\beta}(1 - s_{\beta-\alpha}) + c^2_{\beta-\alpha}(c_{3\beta}s_{\beta-\alpha} - s_{3\beta}c_{\beta-\alpha})\right)\right. \\
&+ \left. 2\left(s_\alpha(s^2_\beta M_A^2 - c^2_\alpha M_H^2) - c_\alpha s_{2\beta}M^2_{H\pm}\right)\right\}
\end{aligned}$$

Compared to the previous parametrization, this looks rather more complicated as it involves the charged Higgs mass as well as the heavy H beside the pseudoscalar Higss mass. Nonethless all these masses are screened.

Dim-4 operators and independent parameters

To easily understand our finding about the screening and the number of independent parameters in the tri-linear and quadrilinear couplings, note that the tri-linear and quadri-linear couplings originate from the quartic terms λ_i only, whereas the mass terms get an additional contribution from the bi-linear terms m_1, m_2, m_{12} . Take for instance the case of the neutral couplings. The quartic self-couplings emerge as combinations of terms in the original fields (before diagonalisation) of the form

$$h_1^4\ ,\ h_2^4\ ,\ h_1^3 h_2\ ,\ h_1^2 h_2^2\ ,\ h_1 h_2^3$$

In terms of the physical scalar fields, h, H:

$$h_1 = -h s_\alpha + H c_\alpha \quad h_2 = h c_\alpha + H s_\alpha$$

which we more judiciously write as

$$\begin{aligned} h_1 &= c_\beta(s_{\beta-\alpha} h + c_{\beta-\alpha} H) + s_\beta(H s_{\beta-\alpha} - h c_{\beta-\alpha}) = c_\beta h_- + s_\beta h_+ \\ h_2 &= s_\beta h_- - c_\beta h_+ \end{aligned}$$

However keep in mind that h_1 and h_2 always appear in the form

$$\begin{aligned} v_1 + h_1 &= v_1\left((1+\frac{h_-}{v'}) + t_\beta h_+\right) \\ v_2 + h_2 &= v_2\left((1+\frac{h_-}{v'}) - t_\beta^{-1} h_+\right) \end{aligned}$$

with $v' = \sqrt{2}v$, which helps write the doublets as

$$\begin{aligned} H_1 &= v_2 \begin{pmatrix} 1+\frac{1}{v'}((h_- - iG^0) + t_\beta(h_+ + iA)) \\ -G^- + t_\beta H^- \end{pmatrix} \\ H_2 &= v_2 \begin{pmatrix} -(G^+ - t_\beta^{-1} H^+) \\ 1+\frac{1}{v'}((h_- + iG^0) - t_\beta^{-1}(h_+ - iA)) \end{pmatrix} \end{aligned}$$

where $G^{\pm,0}$ stand for the Goldstone bosons.

Therefore in effect the quartic terms originate from a combination

$$(1+\frac{h_-}{v'})^2\left(q_1(1+\frac{h_-}{v'})^2 + q_2(1+\frac{h_-}{v'})h_+ + q_3\frac{h_+^2}{v'^2}\right) + q_4(1+\frac{h_-}{v'})\frac{h_+^3}{v'^3} + q_5\frac{h_+^4}{v'^4}$$

While the bi-linear terms are of the form

$$b_1(1+\frac{h_-}{v'})^2 \;+\; b_2(1+\frac{h_-}{v'})\frac{h_+}{v'} \;+\; b_3\frac{h_+^2}{v'^2}$$

Imposing that no tadpole remains (no linear term in h_+, h_-) means that (b_1, q_1) and (b_2, q_2) combine such that one has

$$\begin{aligned}4q_1\frac{h_-^2}{v'^2}(1+\frac{h_-}{v'}+\frac{h_-^2}{4v'^2})+2q_2\frac{h_+h_-}{v'^2}(1+\frac{3h_-}{v'}+\frac{h_-^2}{2v'^2})\\+\frac{h_+^2}{v'^2}(b_3+q_3(1+\frac{h_-}{v'})^2)+q_4(1+\frac{h_-}{v'})\frac{h_+^3}{v'^3}+q_5\frac{h_+^4}{v'^4}\end{aligned}$$

since the coefficients of $h_-^2\,(q_1), h_-h_+(q_2)$ are expressed in terms of masses so do those of h_-^3, h_-^4 as well as $h_+h_-^2, h_+h_-^3$, their accompanying terms in the above can be expressed solely in terms of masses. This does not apply to the tri-linear and quadri-linear terms involving h_+^2. Thus for the tri-linear terms there are only the two structures $h_+^2h_-$ and h_+^3 that can not be expressed solely in terms of the physical masses. The last parameter ϵ only appear in the quartic couplings in the form h_+^4.

Rewriting the Higgs masses in terms of h_- and h_+ will give us the coefficients of h_-^2, h_-^3, h_-^4 and $h_+h_-, h_+h_-^2, h_+h_-^3$.

$$\begin{aligned}&M_h^2h^2+M_H^2H^2 \rightarrow\\&h_-^2(M_h^2s_{\beta-\alpha}^2+M_H^2c_{\beta-\alpha}^2) \;+\; 2h_+h_-s_{\beta-\alpha}c_{\beta-\alpha}(M_H^2-M_h^2)+h_+^2(M_H^2s_{\beta-\alpha}^2+M_h^2c_{\beta-\alpha}^2)\end{aligned}$$

As for the λ dependence it can also be easily derived by just going back to the quartic couplings in terms of the fields h_1, h_2.

$$Q_h=\frac{\lambda_1}{2}h_1^4+\frac{\lambda_2}{2}h_2^4+(\lambda_3+\lambda_4+\lambda_5)h_1^2h_2^2-2\lambda_6h_1^3h_2-2\lambda_7h_2^3h_1$$

when expressed in terms of h_- and h_+ and after moving to the mass basis with for example $\lambda_{5,6,7}$ as extra parameters we get

$$\begin{aligned}(h_+^2+A^2+2H^+H^-)\Big[2(\lambda_ah_--\lambda_bh_+)+\frac{1}{v'}\Big(\lambda_a(h_-^2+G^{0^2}+2G^+G^-)\\-2\lambda_b(h_-h_+-AG^0-(H^+G^-+H^-G^+))+\tilde{\lambda}_c(h_+^2+A^2+2H^+H^-)\Big)\Big]\end{aligned}$$

We see that by completing the h_-, h_+ dependence of $H_{1,2}$ we even get the full tri-linear and quadri-linear couplings involving the pseudo-scalar, charged and Goldstone Boson couplings. It is not hard to see that by re-expressing h_- and h_+ in terms of h and H that we recover all our results. Moreover this writing immediately shows that tri-linear scalar couplings involving Goldstone Bosons can all be expressed most simply in terms of the physical Higgs masses only. The requirement for the absence of tadpoles is a crucial one and explains most of our findings when restricting ourselves to dim-4 operators.

Radiative Corrections

Our results can also be exploited for easily expressing the radiative corrections to the tri-linear (and for that matter quadri-linear) couplings of the SUSY Higgses and explain some of the properties pointed in the literature. Three-point one-loop radiative corrections for the neutral Higgs system in the MSSM have been calculated within the effective potential approximation (Barger et al., Pandita-Osland) The one-loop radiative corrections to both the tri-linear, λ_{hhh} and quadrilinear λ_{hhhh} lightest Higgs self couplings have been revisited recently (Hollik Nov 01). For the case of no mixing in the stop sector it is shown analytically (Hollik) that the bulk of the corrections in the couplings are absorbed by using the corrected Higgs mass while the same is demonstrated numerically for the case of large mixing. One-loop radiative corrections for λ_{hAA} and λ_{HAA} are also considered (Brignole), here also the corrections are found to be large before re-expressing the results in terms of the corrected masses.

The paper of Carena et al., gives analytical approximations (including 2-loop leading-log corrections) for the effective quartic couplings λ_i using a renormalisation-group improved leading-log approximation. We can adapt their formulae to the one-loop case with large stop masses so that we can compare with the direct calculation of the vertices performed by say Barger et al.,

For instance, we find the leading one-loop contributions in the limit where the SUSY breaking term, M_S^2, defined below, is (much) larger than the top mass, (which is consistent with our approach of keeping only the dimension-4 operators)

$$\begin{aligned}
\Delta\lambda_{hhh} &= -\frac{g^4 m_t^4}{64\pi^2 M_w^4}\frac{c_\alpha^3}{s_\beta^3}\left\{6\log(M_S^2/m_t^2) \;+3\frac{f_t(c_t+f_t)}{M_S^2} - \frac{c_t f_t^3}{2M_S^4}\right\} \\
c_t &= A_t + \mu/t_\beta \quad e_t = A_t + \mu/t_\alpha \quad f_t = A_t - \mu t_\alpha \\
m^2_{\tilde{t}_1,\tilde{t}_2} &= M_S^2 \pm m_t c_t
\end{aligned}$$

These shifts correct the tree-level expressions. In the limit $\mu \to 0$, all the λ_i in the MSSM vanish but λ_2.

$$\begin{aligned}
\lambda_2^{\tilde{t}_1,\tilde{t}_2} &= \frac{3}{32\pi^2}\frac{g^4 m_t^4}{s_\beta^4 M_W^4}\left\{\log(M_S^2/m_t^2) + \frac{A_t^2}{M_S^2}(1 - A_t^2/12M_S^4)\right\} \sim .2 \\
&\text{for} \quad A_t, M_S \sim 1TeV
\end{aligned}$$

while in the same approximation

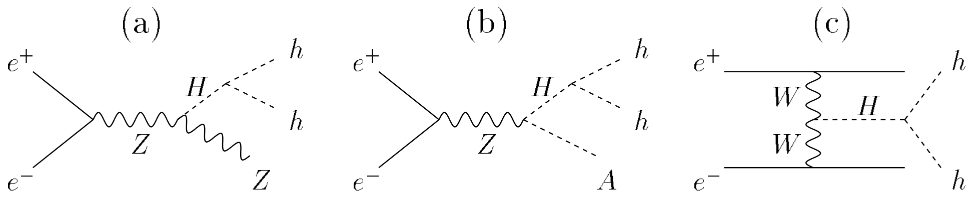
$$
\begin{aligned}
M_h^2 &= M_Z^2 s_{\beta+\alpha}^2 + M_A^2 c_{\beta-\alpha}^2 + \delta M_h^2 \\
\delta M_h^2 &= \frac{3}{4\pi^2}\frac{m_t^4}{v'^2}\left(\log(M_S^2/m_t^2)\ + \frac{f_t c_t +}{M_S^2} - \frac{c_t^2 f_t^2}{12 M_S^4}\right) \\
&\times \left(1 + \frac{c_{\beta-\alpha}}{s_\beta^2}(s_{2\beta}s_{\beta-\alpha} + c_{2\beta}c_{\beta-\alpha})\right)
\end{aligned}
$$

so that one recovers the decoupling property and the fact that the bulk of the radiative corrections are reabsorbed by using the corrected Higgs mass,

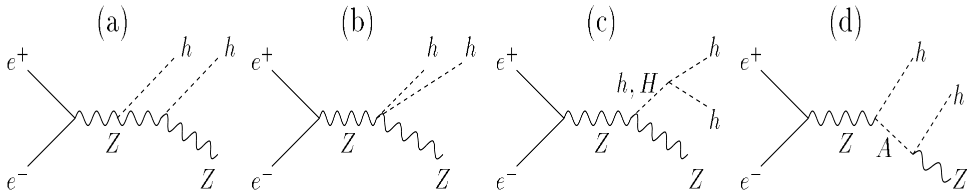
$$
\Delta\lambda_{hhh} = -\frac{c_\alpha}{s_\beta^5}\frac{\delta M_h^2}{12v^2}\ + \ \frac{3g^4 m_t^4}{64\pi^2 M_w^4}\frac{\mu f_t}{M_S^2}\left(1 + \frac{c_t f_t}{6M_S^2}\right)\frac{c_\alpha^2}{s_\beta^4}c_{\beta-\alpha}
$$

Measurements at e^+e^-

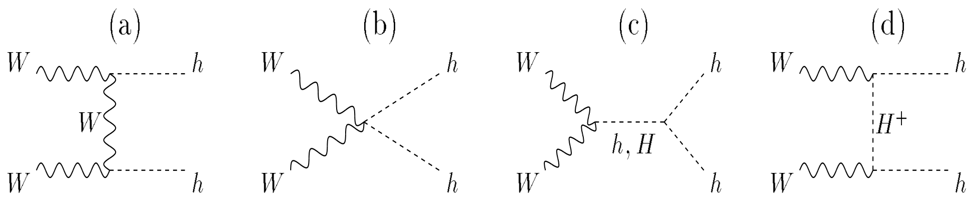
Production Mechanisms



“ SIGNAL”



“Double Strahlung”



FUSION

λ	double Higgsstrahlung (or WW fusion)				triple Higgs−production			
λ	Zhh	ZHh	ZHH	ZAA				
λ	$(\nu_e\bar{\nu}_e hh)$	$(\nu_e\bar{\nu}_e Hh)$	$(\nu_e\nu_e\bar{H}H)$	$(\nu_e\bar{\nu}_e AA)$	Ahh	AHh	AHH	AAA
hhh	× •				×			
Hhh	×	× •			× •	×		
HHh		×	×			×	× •	
HHH			×				× •	
hAA				× •	×	× •		×
HAA				×		×	× •	× •

• indicates which coupling is best measured in the decoupling limit. As the table shows a single process involves more than one coupling, therefore some care is needed to disentangle the various contributions.

As we have seen the measurement of the entire set of the dim-4 operators which is necessary to reconstruct the Higgs potential in SUSY (and 2 Higgs doublet in general) requires that one crosses the thresholds for the production of 3 Higgs bosons, which is not an easy task especially that the cross sections will get tinier and tinier as the Higgs multiplicity increases. As we have seen also, a precise measurement of the Higgs masses is an important ingredient in the reconstruction of this potential. The LHC can thus give a first hint on the parameters λ_i. For instance imagine that the LHC discovers some SUSY particles and identifies them as such but that one discovers also that the lightest Higgs has a mass in excess of 150GeV. This would point to a scalar potential with "hard" λ terms. We could probably even set a rough bound on their possible values. A LC with just enough energy to produce the lightest Higgs and good luminosity would be an ideal machine though.

The different couplings and where to search for them.

Stage 1

Imagine a situation where no heavy Higgs has been produced at a first stage of a linear collider at 500GeV or the LHC, we would then be in the decoupling limit. The only tri-linear couplings which may be accessed are hhh and Hhh through $e^+e^- \to Zhh$ (fusion channels are not efficient at these energies and Higgs masses). However there is no sensitivity to Hhh. Indeed the amplitude for $e^+e^- \to Zhh$, in the unitary gauge, can be written as

$$\mathcal{M}_{Zhh} = a_h \lambda_{hhh} s_{\beta-\alpha} + a_H \lambda_{Hhh} c_{\beta-\alpha} + R$$

where R stands for other contributions not containing the tri-linear Higgs couplings. We have seen that λ_{Hhh} is screened by a factor $c_{\beta-\alpha}$, it is further screened by another such factor when we consider its contribution to this cross section.

At this stage the best would be to reconstruct as precisely as possible m_h and the couplings of h to fermions and the vector bosons. This will help give a bound on the λ and if one makes some model dependent assumptions on the λ_i can be used to extract some information on M_A.

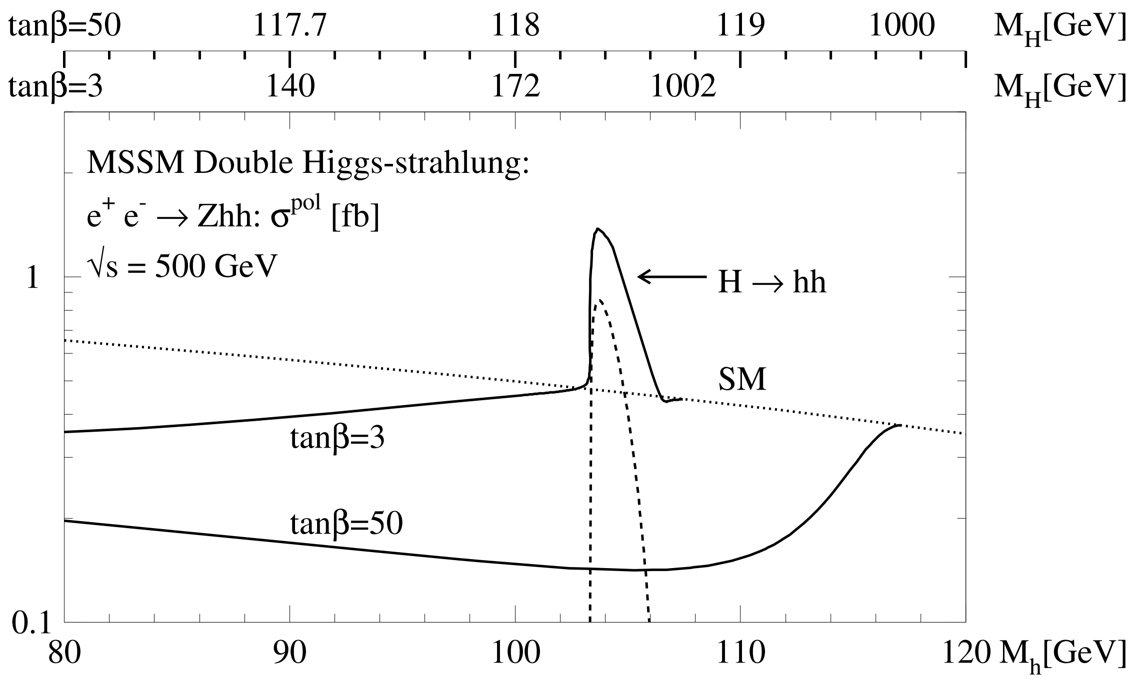
If, on the other hand, the mixing angle is such that $c_{\beta-\alpha}$ is not too small and that M_A is not too large, $e^+e^- \to ZH$ may be accessible. Then the coupling λ_{Hhh} could be reached directly through $H \to hh$. This may still turn out not to be too helpful, since we have seen that the λ_i are still screened in this coupling. Moreover H decays into other particles/superparticles may still be dominant so that $Br(H \to hh)$ will be poorly determined. Let us remark at this point that most of the nice analyses of the SUSY Higgs self-couplings that have been performed were done solely in the context of the minimal supersymmetric model, with no additional "hard" terms in the potential, and have relied heavily on the extremely good precision of the measurement of the dominant branching ratio into $b\bar{b}$. In case h is heavier than 150GeV these analyses need to be extended.

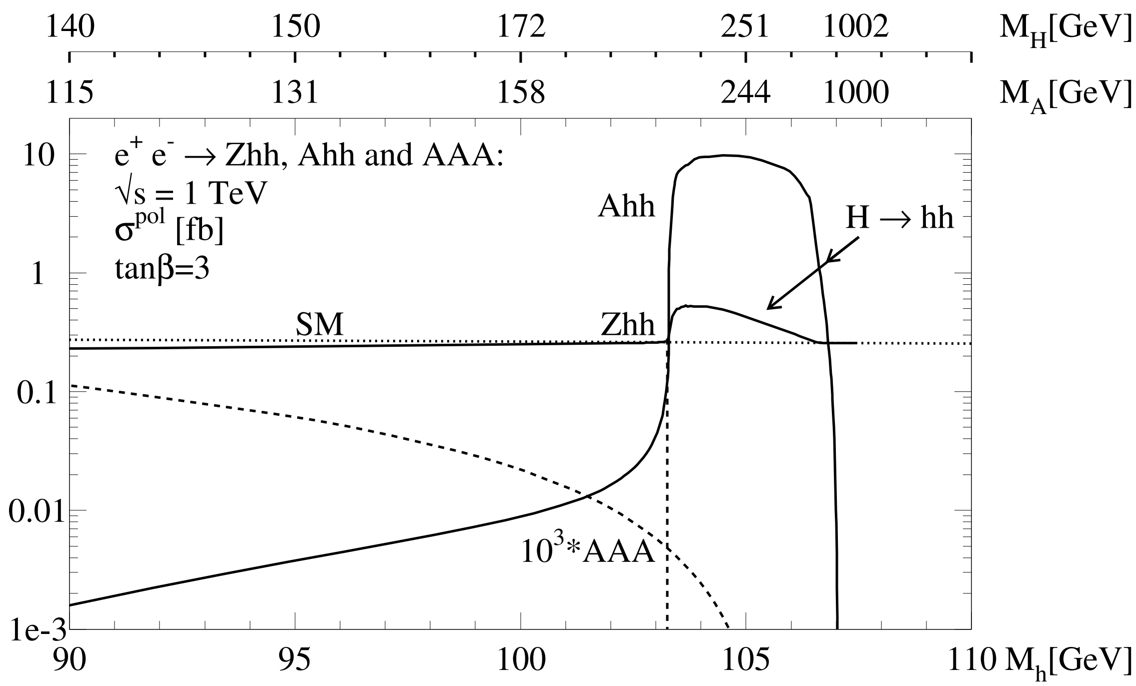
Stage 2

For a machine with higher energies were H and A and thus most probably $H^{\pm}$ have been discovered, the first thresholds for double Higgs production (after that of Zhh) may be

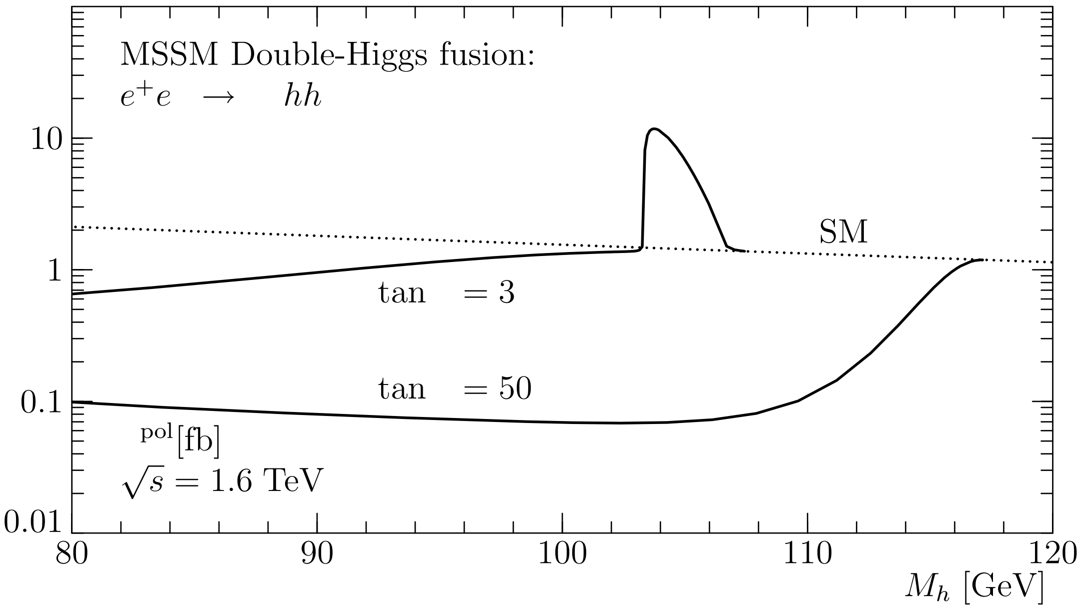
$$
\begin{aligned}
e^+e^- &\to ZhH \qquad \mathcal{M}_{ZHh} = b_h \lambda_{Hhh} s_{\beta-\alpha} \ + \ b_H \lambda_{HHh} c_{\beta-\alpha} + R \\
e^+e^- &\to hhA \qquad \mathcal{M}_{Ahh} = c_h \lambda_{hhh} c_{\beta-\alpha} \ + \ c_H \lambda_{Hhh} s_{\beta-\alpha} + c_A \lambda_{hAA} c_{\beta-\alpha} + R
\end{aligned}
$$

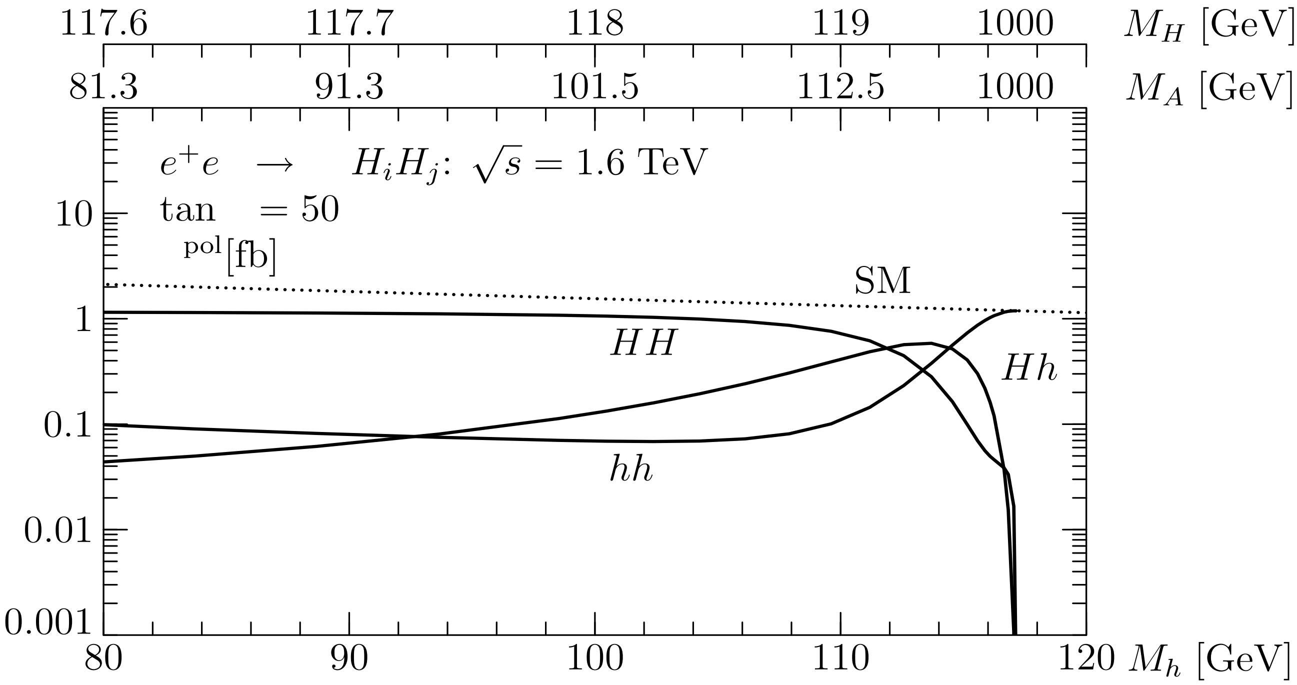
Again, unfortunately these two reaction will not be very sensitive to deviations in the tri-linear couplings if one takes into account the screening effect in λ_{Hhh}. Fusion processes could also be exploited at this stage and the next, but they exhibit a similar behaviour to the annihilation processes as far the extraction of most of the self-couplings.





As expected cross sections are always much smaller than those in the $\mathcal{SM}$ expect when $H \to hh$ is open and substantial. As M_A/M_H gets large only Zhh in the continuum remains and approaches the $\mathcal{SM}$. At higher energies one can exploit AH production more efficiently when $H \to hh$ is open. (Muhlleitner)





Some cross sections through fusion (Muhlleitner)

Stage 3

With higher energies one produces two heavy Higgses in association with a light Higgs or a Z.

$$
\begin{aligned}
e^+e^- &\to ZHH \qquad && \mathcal{M}_{ZHH} = d_h \lambda_{HHh} s_{\beta-\alpha} \; + \; d_H \lambda_{HHH} c_{\beta-\alpha} + R \\
e^+e^- &\to ZAA \qquad && \mathcal{M}_{ZAA} = e_h \lambda_{hAA} s_{\beta-\alpha} \; + \; e_H \lambda_{HAA} s_{\beta-\alpha} + R \\
e^+e^- &\to hHA \qquad && \mathcal{M}_{AHh} = f_h \lambda_{HHh} s_{\beta-\alpha} \; + \; f_H \lambda_{hAA} s_{\beta-\alpha} + f'_h \lambda_{Hhh} + \\
e^+e^- &\to H^+H^-h \qquad && \mathcal{M}_{H^+H^-h} = g_h \lambda_{H^+H^-h} + R
\end{aligned}
$$

As can be seen all of these reactions will be used to determine λ_a (λ_b will still be screened).

Stage 4

At even higher energies, production of three heavy Higgs bosons will allow to determine λ_b. The processes at our disposal will be

$$
\begin{aligned}
e^+e^- &\to AAA & \mathcal{M}_{AAA} &= i_h\lambda_{hAA}c_{\beta-\alpha} \;+\; i_H\lambda_{HAA}s_{\beta-\alpha} \\
e^+e^- &\to HHA & \mathcal{M}_{HHA} &= j_h\lambda_{hHH}c_{\beta-\alpha} \;+\; j_H\lambda_{HHH}s_{\beta-\alpha} + j_A\lambda_{HA\ldots} \\
e^+e^- &\to H^+H^-H & \mathcal{M}_{H^+H^-H} &= k_H\lambda_{H^+H^-H} + R \\
e^+e^- &\to H^+H-A & \mathcal{M}_{H^+H^-A} &= l_hc_{\beta-\alpha}\lambda_{H^+H^-h} + l_Hs_{\beta-\alpha}\lambda_{H^+H^-h}
\end{aligned}
$$

Stage 5

Production of four heavy Higgses, like $e^+e^- \to HHHH, H^+H^-H^+, H^-,$ will help measure λ_c. Ideally one thus needs a multi-TeV, high-luminosity machine to achieve such measurements, however even in this case we do not expect to reach precision better than 10% on the extraction of the couplings.

$\boxed{\text{Examples with } \sqrt{s} = 3TeV}$

parameters: $M_A = 400\text{GeV}$, $M_H = 401\text{GeV}$, $M_{H^\pm} = 408\text{GeV}$, $t_\beta = 10$

$hhA, hHA, hH^+H^-, HH^+H^-, HHA, H^+H^-A, AAA < 10^{-4}\text{fb}$

ZH^+H^-, ZHH, ZAA, ZHA of order 10^{-2}fb any hope??

Effects from higher order operators

Up to now we have only discussed the effects of the dim-4 operators.Higher order operators are doomed to contribute less significantly, as their effects are explicitly screened by a high scale. We will illustrate this case by considering only three new operators, and restrict ourselves to a few Higgs self couplings to make the point. We consider

$$V_{eff} \to V_{eff} + \frac{1}{\Lambda^2}\left\{\tilde{\kappa}_1(H_1H_1^*)^3 + \tilde{\kappa}_2(H_2H_2^*)^3 + \tilde{\kappa}_3(H_1H_1^*)^2(H_2H_2^*)\right\}$$

For notational ease we will use

$$\kappa_i = \frac{v'^2}{\Lambda^2}\tilde{\kappa}_i$$

with Λ the scale of new physics. We find

$$\begin{aligned}
\lambda_{hhh} &= (\lambda_1 - 6\kappa_1 c_\beta^2 - 2\kappa_3 s_\beta^2)\; c_\beta s_\alpha^3 - (\lambda_2 - 6\kappa_2 s_\beta^2)\; c_\alpha^3 s_\beta \\
&+ \frac{1}{2}((\lambda_3 + \lambda_4 - 2\kappa_3 c_\beta^2) + \lambda_5)\; s_{2\alpha} c_{\beta+\alpha} \\
&\quad +\lambda_6 s_\alpha^2 (c_{\beta+\alpha} + 2c_\alpha c_\beta) + \lambda_7 c_\alpha^2 (c_{\beta+\alpha} - 2s_\alpha s_\beta) \\
&\quad -4\kappa_1 c_\beta^3 s_\alpha^3 + 4\kappa_2 s_\beta^3 c_\alpha^3 + 4 s_\beta c_\beta^2 c_\alpha s_\alpha^2 \kappa_3
\end{aligned}$$

where we have split the κ_i contribution in such a way that the first contribution is just a shift in the λ_i contributions which would be re-absorbed when expressed in terms of masses.

Indeed, by using masses and keeping as extra parameters $\lambda_{5,6,7}$ and $\kappa_{1,2,3}$ we get

$$
\begin{aligned}
\lambda_{hhh} &= \frac{1}{2v^2}\frac{m_h^2}{s_\beta c_\beta}(s_\beta s_\alpha^3 - c_\alpha^3 c_\beta) \\
&+ \frac{c_{\beta-\alpha}^2}{s_\beta c_\beta}\left(\lambda_5 c_{\beta+\alpha} + \lambda_6 c_\beta s_\alpha - \lambda_7 s_\beta c_\alpha + \frac{M_A^2}{2v^2} c_{\beta+\alpha}\right) \\
&-\frac{1}{2}s_{2\beta}^3\left(\frac{s_\alpha^3}{s_\beta^3}\kappa_1 + \frac{c_\alpha^3}{c_\beta^3}\kappa_2 + \frac{s_\alpha^2}{s_\beta^2}\frac{c_\alpha}{c_\beta}\kappa_3\right)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{Hhh} &= c_{\beta-\alpha}\left\{\lambda_5(\frac{1}{3} - \frac{s_\alpha c_\alpha}{s_\beta c_\beta}) + \lambda_6(\frac{c_\beta}{3s_\beta} + \frac{s_\alpha}{s_\beta}s_{\beta-\alpha}) + \lambda_7(\frac{s_\beta}{3c_\beta} - \frac{c_\alpha}{c_\beta}s_{\beta-\alpha})\right. \\
&+ \left.\frac{1}{6v^2}\left(\frac{s_\alpha c_\alpha}{s_\beta c_\beta}(2M_h^2 + (M_H^2 - M_A^2)) + M_A^2\right)\right\} \\
&+4\left(c_\beta^3 s_\alpha^2 c_\alpha \kappa_1 + s_\beta^3 c_\alpha^2 s_\alpha \kappa_2 + (\frac{2}{3} - s_\alpha^2)c_\beta^2 s_\beta s_\alpha \kappa_3\right)
\end{aligned}
$$

We see that the higher order operators are not further reduced by the decoupling factor $c_{\beta-\alpha}$ and that all κ_i contribute to all the self-couplings, unlike with λ_i where we are only left with a combination of two couplings. This means that if one ideally have measured all the masses and couplings to ordinary fermions and quite precisely all the tri-linear Higgs self-couplings one could tell whether higher order operators are contributing. However considering the foreseen precision on the extraction of the Higgs self-couplings and the expected small contribution of the higher order terms, this would seem to be over optimistic.

Conclusions

- **If the scalar Higgs is discovered one needs to precisely probe its properties.**
- **The measurement of the self-couplings essential for a reconstruction of the scalar potential**
- **These measurements are extremely difficult and need the highest energy and luminosity**
- **the quartic couplings may never be measured**
- **SUSY breaking, EW symmetry breaking and the Higgs potential. Have we been too conventional and dogmatic?**
- **There is a screening of the leading contribution to the self-couplings of the Higgs once the mass of the lightest is measured.**
- **However measurements of the masses of the Higgses and their couplings to matter and gauge bosons may provide important clues.**