1 General Formalism and Approximations

Assuming a stable Dark Matter particle (S) weakly coupled to the SM through a single mediator (M), the Boltzmann equation for the DM number density can be approximately written as:

$$\frac{dn_S}{dt} + 3Hn_S = \left(\bar{n}_S^2 - n_S^2\right)\langle\sigma v\rangle + \frac{\Gamma_M}{\gamma_M}\left(n_M - \bar{n}_M \frac{n_S}{\bar{n}_S}\right) \tag{1}$$

where Γ_M is the width for $M \to S + X$ decays, $\bar{n}_i$ is the equilibrium density and γ_M is the relativistic dilution factor, given by:

$$\frac{1}{\gamma_M} = m_M \langle \frac{1}{E_M} \rangle \tag{2}$$

The first term on the RHS of Eq.(1) corresponds to DM production/annihilation through $2 \to 2$ scatterings, while the second term corresponds to production/annihilation through decays or inverse decays of the mediator M. The Boltzmann equations for the mediator can be written as:

$$\frac{dn_M}{dt} + 3Hn_M = \left(\bar{n}_M^2 - n_M^2\right)\langle\sigma v\rangle - \frac{\Gamma_M}{\gamma_M}\left(n_M - \bar{n}_M \frac{\bar{n}_S}{n_S}\right) \tag{3}$$

In order to obtain the DM relic abundance, Eqs.(1) and (3) must be integrated from the initial conditions (at reheating) until today. Furthermore, additional equations must be included for the evolution of entropy (in case it is not conserved) and the energy densities of S and M. Below we briefly discuss how these can be solved numerically under minimal assumptions and then how an analytical solution can be found if we assume entropy conservation, a radiation-dominated universe and a mediator always in thermal equilibrium.

1.1 Numerical Solution

One of the main difficulties in solving Eqs.(1) and (3) is the proper evaluation of the dilution factor γ_M, since this depends on the average energy of the mediator. In the simple scenario where the mediator is always in thermal equilibrium this factor can be easily computed and gives:

$$\frac{1}{\gamma_M} = \frac{1}{\bar{n}_M}\frac{m_M^2 T}{2\pi^2} K_1\left(m_M/T\right) \tag{4}$$

where $K_1(x)$ is the modified Bessel function of the second kind of order 1. However, in general this is not true if the mediator distribution departs from equilibrium. In order to deal with such cases, we make the following approximation:

$$\frac{1}{\gamma_M} = m_M \langle \frac{1}{E_M} \rangle \simeq m_M \frac{n_M}{\rho_M} \tag{5}$$

which corresponds to:

$$K_1(x) \simeq \frac{K_2(x)^2}{K_1(x) + 3K_2(x)/x} \tag{6}$$

if the mediator is in thermal equilibrium. This is a good approximation for $x = m_M/T >> 1$ or in other words, if the mediator decays at temperatures much smaller than its mass.

Finally, a simple equation can be written for $R_M = \rho_M/n_M$:

$$\frac{dR_M}{dt} = -3H\frac{P_M}{n_M} \tag{7}$$

where P_M is the pressure of M: $P_M = \rho_M/3$ (0) for a relativistic (non-relativistic) fluid. For the relativistic/non-relativistic transition an interpolation for P_M is used.

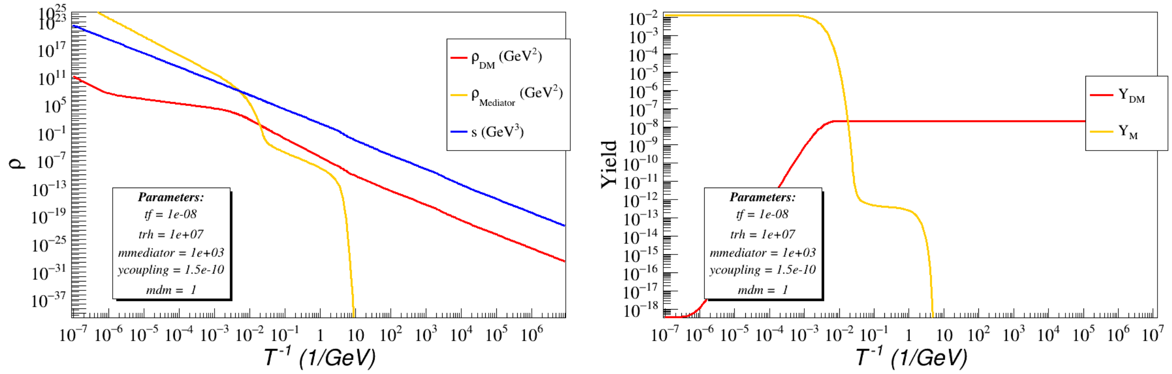

Figure 1: *Left*: Evolution of the mediator and DM energy densities (ρ_M and ρ_{DM}) as a function of the inverse of temperature. The evolution of the entropy density (s) is also shown. *Right:* Evolution of the mediator and DM yields as a function of the inverse of temperature. The parameters shown refer to the simplified FIMP scenario described in Sec.2.

Under the above approximations, the coupled Boltzmann equations can be solved once we include the equation for the entropy:

$$\frac{dS}{dt} = \frac{a^3}{T}\Gamma_{MX} m_M \left(n_M - \bar{n}_M \frac{n_S}{\bar{n}_S} \right) \tag{8}$$

where a is the scale factor, $\Gamma_{MX} = \Gamma_M f_x$ and f_x is the fraction of energy injected in the thermal bath from the mediator decays. Typically, for a two body decay ($M \to S + X$), we have $f_x = 1/2$. Unless the mediator number density is very large ($n_M \gg \bar{n}_M$), the RHS of the above equation is approximately zero and entropy is conserved.

The numerical solution of Eqs.(1),(3),(7) and (8) for a specific choice of parameters is shown in Fig.1.

1.2 Analytical Solution

In order to analytically solve Eq.(1), we make the following assumptions:

- a weakly coupled DM, so $n_S/\bar{n}_S \ll 1$ and $\langle \sigma v \rangle \bar{n}_S^2 \simeq 0$;
- a strongly coupled mediator, so $n_M = \bar{n}_M$ and γ_M is given by Eq.(4);
- a radiation dominated universe, so $H = \sqrt{\frac{8\pi^3}{90}\frac{g_*(T)T^4}{M_P^2}}$;
- entropy conservation ($\dot{S} = 0$).

Under the above assumptions Eq.(1) simplifies to:

$$\frac{dn_S}{dt} + 3Hn_S \simeq \frac{\Gamma_M}{\gamma_M} n_M \tag{9}$$

which, together with entropy conservation, leads to the following equation for the yield ($Y_S = n_S/s$):

$$\frac{dY_S}{dt} \simeq \frac{\Gamma_M}{\gamma_M} Y_M \tag{10}$$

Using $d/dt = -HTd/dT$, we have:

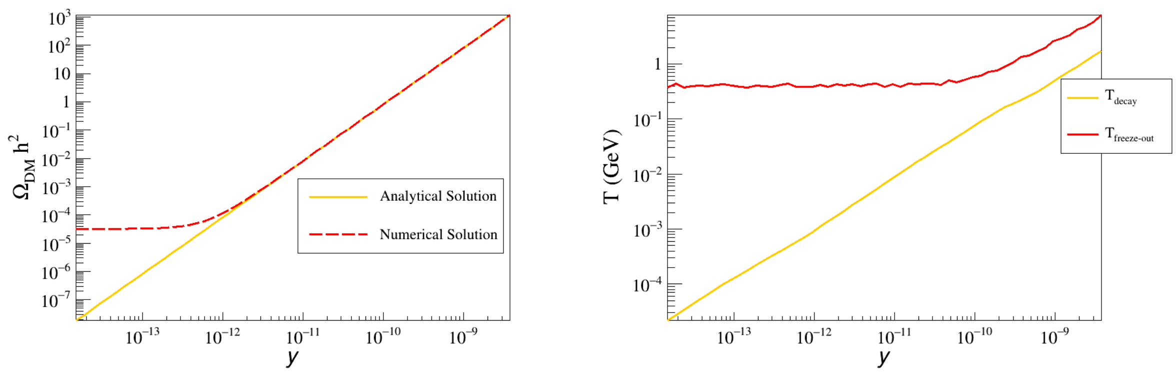

Figure 2: *Left*: The relic DM abundance computed using Eq.(13) and the numerical solution of the Boltzmann equations (see Sec.1.1) as a function of the mediator-DM coupling y. *Right:* Approximate values for the mediator freeze-out and decay temperatures (obtained numerically) as a a function of the mediator-DM coupling y. The parameters used are the same as in Fig.1 and correspond to the simplified FIMP scenario described in Sec.2.

$$\frac{dY_S}{dT} = -\frac{\Gamma_M}{\gamma_M}\frac{Y_M}{HT} \Rightarrow Y_S = \Gamma_M \int_0^{T_{RH}} \frac{1}{\gamma_M}\frac{Y_M}{HT} dT \tag{11}$$

Finally, using the assumption that the mediator is always in thermal equilibrium:

$$\frac{Y_M}{\gamma_M} = \frac{1}{s}\frac{m_M^2 T}{2\pi^2} K_1\left(m_M/T\right) \tag{12}$$

and defining $x = m_M/T$, we obtain:

$$Y_S = \left(\frac{45}{4\pi^4}\frac{M_P}{1.66}\right) g_M \frac{\Gamma_M}{m_M^2}\int_{m_M/T_{RH}}^{\infty} \frac{K_1(x)x^3}{\sqrt{g_*(m_M/x)}g_*^S(m_M/x)} dx \tag{13}$$

where we used $s = \frac{2\pi^2}{45} g_*^S(T)T^3$.

If we further assume that $g_*(m_M/x), g_*^S(m_M/x) \simeq g_*(m_M), g_*^S(m_M)$ and $T_{RH} \gg m_M$, the integral can be evaluated, giving:

$$Y_S = \left(\frac{135}{8\pi^3}\frac{M_P}{1.66}\right) g_M \frac{\Gamma_M}{m_M^2}\frac{4.71}{\sqrt{g_*(m_M)}g_*^S(m_M)} \tag{14}$$

In Fig.2 (left) we show the relic DM abundance as a function of the DM-mediator coupling (y) obtained using Eq.(13). Since the mediator width increases with y ($\Gamma_M \propto y^2$), the relic abundance increases quadratically with y, as seen in the plot. We also show the result obtained using the numerical integration discussed in Sec.1.1. As we can see both results agree very well, except for very low values of y. At these values the mediator is extremely long-lived and it decouples from the thermal bath before decaying. As a result, the approximation that M is always in thermal equilibrium ($Y_M = \bar{Y}_M$) is not valid and the result obtained in Eq.(13) no longer applies. As seen in Fig.2 (right) this happens when $T_{decay} \ll T_{freeze-out}$. For such cases the DM relic abundance is simply given by $\Omega_{DM}h^2 = \frac{m_{DM}}{m_M}\Omega_M h^2$, where $\Omega_M h^2$ is the mediator relic abundance before its decay. Furthermore, since $\Omega_M h^2$ does not depend on y, the DM relic abundance also becomes y-independent as shown by the numerical solution in Fig.2 at low y values.

2 Simplified FIMP Scenario

If we define the S-M-SM coupling as:

$$ySMP_{SM} \tag{15}$$

and assume the mediator has gauge couplings to the SM, we have:

$$\Gamma_M = \frac{y^2}{8\pi} m_M,\ \langle \sigma v \rangle_M \simeq \frac{\alpha}{m_M^2},\ \langle \sigma v \rangle_S \simeq \frac{y^4}{m_S^2} \tag{16}$$

and $f_x = 1/2$, where f_x is the fraction of energy injected in the thermal bath coming from $M \to S+X$ decays, which is valid for $m_M \gg m_S$.

With the above definitions we can numerically and analytically compute the relic abundance of DM, following the procedures described in Secs.1.1 and 1.2. In Fig.2 we show the relic abundance as a function of y for both the analytic and the numerical solution.