A Primer on Iterative Subtraction at NNLO

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Subtraction scheme

- Extension of FKS to NNLO by adding *sectors* to separate overlapping singularities.
- [Czakon ‘10, ‘11; Boughezal, Melnikov, Petriello ‘12; Czakon, Heymes ‘14].
- Expect simplification when recombining (as in FKS) – **not apparent** in above formulations.
- Simplified implementation – focus on **gauge-invariant matrix elements**:
  - Independent treatment of soft and collinear singularities.
  - Easier recombination of sectors.
  - Explicit pole cancellation for different kinematic structures.

Wishlist for an NNLO subtraction scheme

- Local.
- Straightforward with clear origin of singularities.
- Explicit (if possible, analytic) cancellation of poles.
- Process-independent.
- Allowing four-dimensional evaluation of matrix elements.
- ...

For simplicity...

- Focus on color singlet final state.
- Treatment of colored final states conceptually the same .
- Proof-of-concept: DY (exact results known [Hamberg, Matsuura, van Neerven '91]).
  - All partonic processes checked.
- Can implement new processes relatively quickly.
- Discuss $q\bar{q} \to V + n\ g$:
  - Most complicated singular structure.
  - Other partonic channels are simplifications of this.

Subtraction at NLO

$$\mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = \mathrm{d}\sigma^{\mathrm{V}} + \mathrm{d}\sigma^{\mathrm{R}} + \mathrm{d}\sigma^{\mathrm{CV}}$$

Focus on real radiation – process $q\bar{q} \to V + g$:

Subtraction @ NLO: **FKS**

$$\mathrm{d}\sigma^{\mathrm{R}} = \frac{1}{2s} \int [\mathrm{d}g_4] F_{LM}(1,2,4) \equiv \langle F_{LM}(1,2,4) \rangle.$$

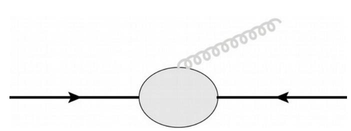
$$F_{LM}(1,2,4) = \mathrm{dLips}_V \; |\mathcal{M}(1,2,4,V)|^2 \; \mathcal{F}_{\mathrm{kin}}(1,2,4,V).$$

$$[\mathrm{d}g_4] = \frac{\mathrm{d}^{d-1} p_4}{(2\pi)^d 2E_4} \theta(E_{\mathrm{max}} - E_4)$$

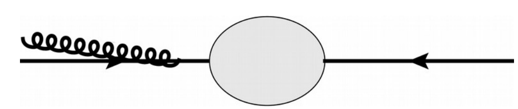
- Lorentz-inv. Phase space for V **(incl. delta-fn)** → dLips_V
- Matrix-element sq. → $|\mathcal{M}(1,2,4,V)|^2$
- IR-safe observable → $\mathcal{F}_{\mathrm{kin}}(1,2,4,V)$
- **Integration in partonic CoM frame** → $[\mathrm{d}g_4]$
- **Arbitrarily large energy parameter** → E_{max}

Singular regions:

- g_4 → soft.



- g_4 → collinear to either initial state parton.



Iterative subtraction

Define operators:

$$S_i A = \lim_{E_i \to 0} A \qquad C_{ij} A = \lim_{\rho_{ij} \to 0} A \qquad \rho_{ij} = 1 - \cos\theta_{ij}$$

Rewrite as

$$\begin{aligned} \langle F_{LM}(1,2,4)\rangle = & \langle S_4 F_{LM}(1,2,4)\rangle + \\ & \langle (C_{41} + C_{42})(I - S_4) F_{LM}(1,2,4)\rangle + \\ & \langle (I - C_{41} - C_{42})(I - S_4) F_{LM}(1,2,4)\rangle . \end{aligned}$$

- Third term: finite, can be integrated numerically in 4-dimensions.
- First term: soft gluon decouples completely → need upper bound: $E_{\max}$.
- Second term: collinear and soft+collinear gluon decouples partially or completely.
- Singularities made explicit by integrating over decoupled gluon.

Collinear limits

The collinear limit is

$$C_{41} F_{LM}(1,2,4) = \frac{g_{s,b}^2}{E_4^2 \rho_{41}} (1-z) P_{qq}(z) \left(\frac{F_{LM}(z \cdot 1, 2)}{z} \right).$$

$$z = 1 - E_4/E_1$$
$$P_{qq}(z) = P_{qq}^{(0)}(z) + \mathcal{O}(\epsilon)$$

- Gluonic *angles* decouple – integrate over them to get $1/\epsilon$ pole.
- Changing $\mathrm{d}E_4 \to \mathrm{d}z$ and then rewriting as **plus-prescription –** explicit singular structure.

$$\langle C_{41} F_{LM}(1,2,4) \rangle = -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \left[-\left(\frac{C_F}{\epsilon} + \frac{3C_F}{2} \right) \langle F_{LM}(1,2) \rangle + \int_0^1 \mathrm{d}z \mathcal{P}_{qq,R}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} \right\rangle \right].$$

Cancels against virtual

Cancels against pdf renorm

Evaluation in four dimensions

After cancelling poles, we can take the $\epsilon \to 0$ limit and compute everything in four dimensions.

$$2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = \left\langle F_{LV}^{\mathrm{fin}}(1,2) + \frac{\alpha_s(\mu)}{2\pi}\left[\frac{2}{3}\pi^2 C_F F_{LM}(1,2)\right]\right\rangle + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,4)\rangle +$$
$$+ \frac{\alpha_s(\mu)}{2\pi}\int_0^1 \mathrm{d}z \left[\ln\frac{s}{\mu^2}\hat{P}_{qq}^{(0)}(z) - \mathcal{P}_{qq,R}^{(\epsilon)}(z)\right]\left\langle \frac{F_{LM}(z\cdot 1,2)}{z} + \frac{F_{LM}(1,z\cdot 2)}{z}\right\rangle .$$

Sum of:

- Lower particle-multiplicity terms, with or without convolutions with splitting functions.
- Real emission term, with singular configurations removed by iterated subtraction.
- Finite remainder of virtual corrections.

NNLO: Real-real Corrections

Process $q\bar{q} \to V + gg$.

$$2s \cdot \mathrm{d}\sigma^{\mathrm{RR}} = \frac{1}{2!} \int [\mathrm{d}g_4][\mathrm{d}g_5] F_{LM}(1,2,4,5).$$

Singularity structure much more complicated than at NLO:

- g_4 or g_5 → soft.
- g_4 or g_5 → collinear to initial state partons.
- g_4 or g_5 → collinear to each other.
- Combination of the above – can approach **each limit in different ways!**

Separating the singularities is the name of the game!

Soft and collinear singularities

BUT: we are dealing with gauge-invariant matrix elements (as opposed to individual Feynman diagrams):

- **Can regulate soft and collinear singularities independently.**
- Order energies $E_4 > E_5$: either double soft ($\mathbb{S}$) or gluon 5 soft.
- Regulate soft singularities:

$$\big\langle F_{LM}(1,2,4,5)\big\rangle = \big\langle \mathbb{S} F_{LM}(1,2,4,5)\big\rangle + \big\langle S_5(I-\mathbb{S})F_{LM}(1,2,4,5)\big\rangle \\ + \big\langle (I-S_5)(I-\mathbb{S})F_{LM}(1,2,4,5)\big\rangle.$$

then regulate collinear singularities in each term

Regulating collinear singularities

- **Step 1:** Introduce **phase-space partitions**

$$1 = w^{14,15} + w^{24,25} + w^{14,25} + w^{15,24}.$$

Triple collinear partitions:

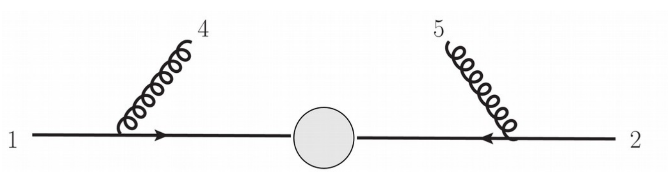
$w^{14,15}$ contains $C_{41},\ C_{51},\ C_{45}$

$w^{24,25}$ contains $C_{42},\ C_{52},\ C_{45}$

Double collinear partitions:

$w^{14,25}$ contains $C_{41},\ C_{52}$

$w^{15,24}$ contains $C_{42},\ C_{51}$



Sector Decomposition

- **Step 2: Sector decomposition:**
- Triple collinear sectors still have **overlapping** singularities.
- Define angular ordering to separate singularities.

$$1 = \theta\left(\eta_{51} < \frac{\eta_{41}}{2}\right) + \theta\left(\frac{\eta_{41}}{2} < \eta_{51} < \eta_{41}\right) \qquad \eta_{ij} = \rho_{ij}/2$$

$$+\,\theta\left(\eta_{41} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{41} < \eta_{51}\right) \equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}.$$

- Thus the limits are $\theta^{(a)} : C_{51}$ $\quad \theta^{(b)} : C_{45}$

  $\theta^{(c)} : C_{41}$ $\quad \theta^{(d)} : C_{45}$

Removing collinear singularities

Then we can write soft regulated term as

$$\Big\langle (I - S_5)(I - \not{S}) F_{LM}(1,2,4,5) \Big\rangle = \Big\langle F_{LM}^{s_r c_s}(1,2,4,5) \Big\rangle + \Big\langle F_{LM}^{s_r c_t}(1,2,4,5) \Big\rangle + \Big\langle F_{LM}^{s_r c_r}(1,2,4,5) \Big\rangle,$$

$\langle F_{LM}^{s_r c_r}(1,2,4,5) \rangle$

- All singularities removed through iterated subtractions – evaluated in 4-dimensions.
- Only term involving fully-resolved real-real matrix element

$\langle F_{LM}^{s_r c_{s,t}}(1,2,4,5) \rangle$

- Contain (soft-regulated) single and triple collinear singularities.
- Matrix elements of lower multiplicity.
- Partitioning factors and sectors: one collinear singularity in each term.

Treating singular limits

We have four singular subtraction terms:

$$\left\langle \mathbb{S} F_{LM}(1,2,4,5)\right\rangle \quad \left\langle S_5 (I-\mathbb{S}) F_{LM}(1,2,4,5)\right\rangle \quad \left\langle F_{LM}^{s_r c_s}(1,2,4,5)\right\rangle \quad \left\langle F_{LM}^{s_r c_t}(1,2,4,5)\right\rangle$$

We know how to treat them:

- Gluon(s) decouple partially or completely.
- Decouple completely:
  - Integrate over gluonic angles and energy.
- Decouple partially:
  - Integrate over gluonic angles.
  - Integral(s) over energy → integrals over splitting function in *z.*
- Results in **lower particle multiplicity terms** convoluted with (new) splitting functions.

Double-collinear partition

In single-collinear subtraction:

$$DC = \Big\langle [I - \not{S}]\,[I - S_5]\Big[(C_{41}[\mathrm{d}g_4] + C_{52}[\mathrm{d}g_5])\, w^{14,25} + (C_{42}[\mathrm{d}g_4] + C_{51}[\mathrm{d}g_5])\, w^{24,15} \Big] \times F_{LM}(1,2,4,5) \Big\rangle .$$

Limit acts on phase space!

Consider first term:

$$\Big\langle [I - \not{S}]\,[I - S_5] C_{41}[\mathrm{d}g_4] w^{14,25} F_{LM}(1,2,4,5) \Big\rangle$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int\limits_0^{z_{\max}(E_5)} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \mathcal{P}_{qq}(z) \Big\langle \tilde{w}^{14,25}_{4||1} [I - S_5] F_{LM}(z\cdot 1, 2, 5) \Big\rangle .$$

$$z_{\max}(E5) \equiv 1 - E_5/E_1 .$$

Ideally: integral on [0:1]

Double-collinear partition

$$DC = \left\langle [I - \mathbb{S}]\,[I - S_5]\left[(C_{41}[\mathrm{d}g_4] + C_{52}[\mathrm{d}g_5])\,w^{14,25} + (C_{42}[\mathrm{d}g_4] + C_{51}[\mathrm{d}g_5])\,w^{24,15}\right]\right.$$

$$\left.\times F_{LM}(1,2,4,5)\right\rangle.$$

Now fourth term:

$$\langle [I - \mathbb{S}]\,[I - S_5]\,C_{51}[\mathrm{d}g_5] w^{24,15} F_{LM}(1,2,4,5)\rangle$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int\limits_{z_{\min}(E_4)}^{1} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \hat{\mathcal{P}}_{qq}^{(-)}(z) \langle \tilde{w}_{5||1}^{24,15} F_{LM}(z \cdot 1, 2, 4)\rangle.$$

$$z_{\max}(E_4) = 1 - E_4/E_1 = z_{\min}(E_4)$$

$$\hat{\mathcal{P}}_{qq}^{(-)} f(z) \equiv \mathcal{P}_{qq}(z) f(z) - 2C_F f(1)$$

Combining partitions

Rename the resolved gluon 4 in the first term and combine:

$$\Big\langle \big[I - \mathbb{S}\big]\big[I - S_5\big]\big[C_{41}[\mathrm{d}g_4]w^{14,25} + C_{51}[\mathrm{d}g_4]w^{15,24}F_{LM}(1,2,4,5)\Big\rangle$$

$$= -\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\int_0^1 \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}}\Big\langle \tilde{w}^{15,24}_{5||1}\bigg(\hat{\mathcal{P}}^{(-)}_{qq}(z)\big[I - S_4\big]F_{LM}(z\cdot 1,2,4) +$$

$$\theta(z_4 - z)2C_F\big[I - S_4\big]F_{LM}(1,2,4) + \theta(z_4 - z)\hat{\mathcal{P}}^{(-)}_{qq}(z)S_4F_{LM}(z\cdot 1,2,4)\bigg)\Big\rangle.$$

Similar simplifications on combining terms from **double** & **triple** collinear partitions.

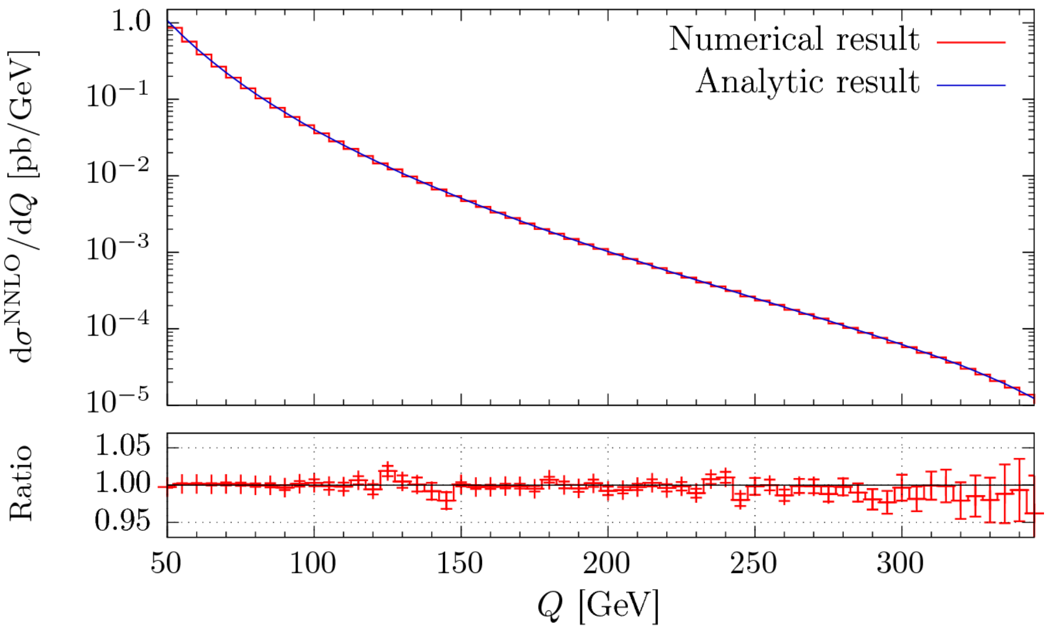
Proof-of-principle

- Calculate $pp \to \gamma^* + X \to e^+e^- + X$ to NNLO.
- Lepton pairs with invariant mass $50 \text{ GeV} \leq Q \leq 350 \text{ GeV}$.
- Extract results from [Hamberg, Matsuura, van Neerven '91] to compare (analytic in *Q).*
- NNLO *contributions* for $q\bar{q} \to \gamma^* + ng$

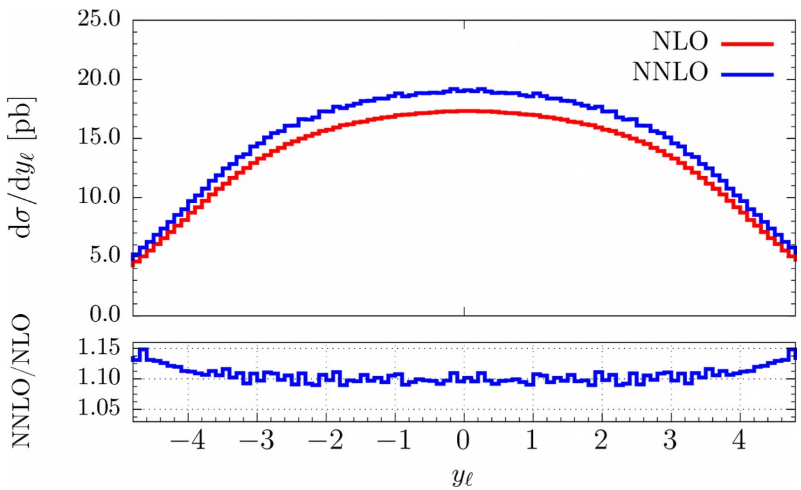
$$\mathrm{d}\sigma^{\mathrm{NNLO}} = 14.471(4) \text{ pb}$$

$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{\mathrm{analytic}} = 14.470 \text{ pb}$$

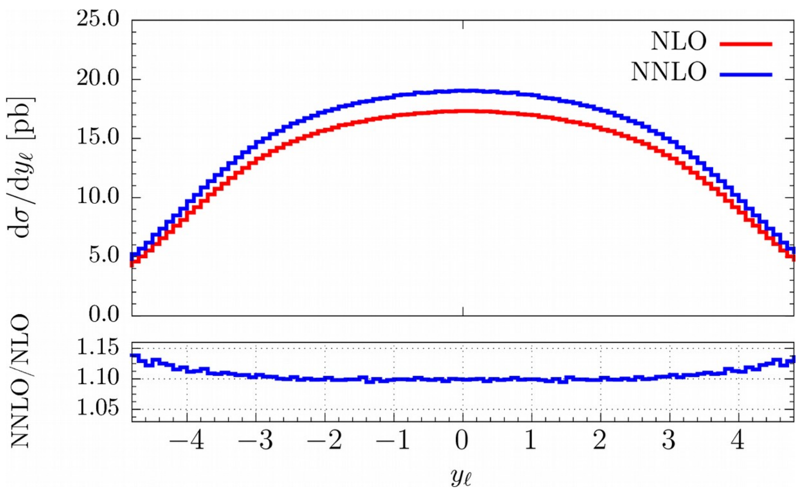
- Sub per-mille agreement in cross sections.
- Per-mille to percent agreement across **5 orders of magnitude** in *Q.*
- Similar level of agreement in **all other partonic channels**.
- Fully differential results.



Differential distributions (I)



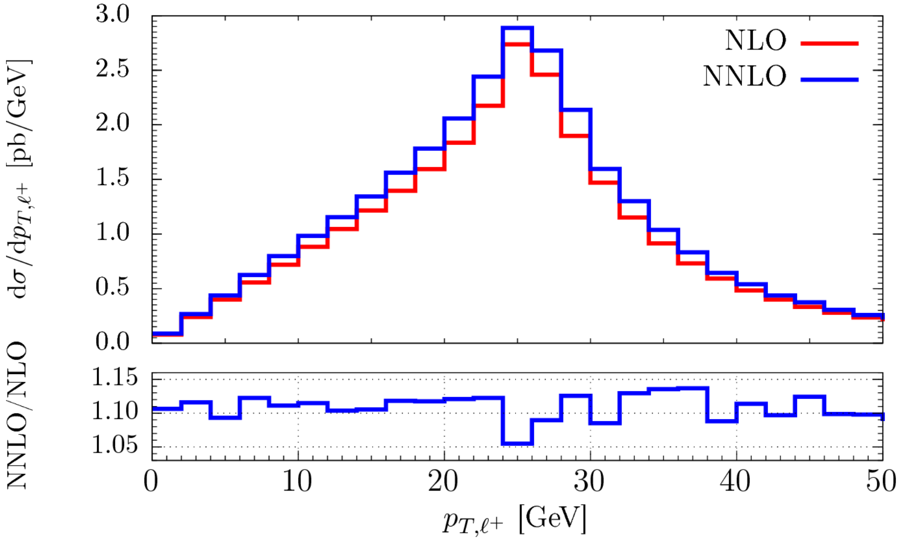
O(10 hours) runtime



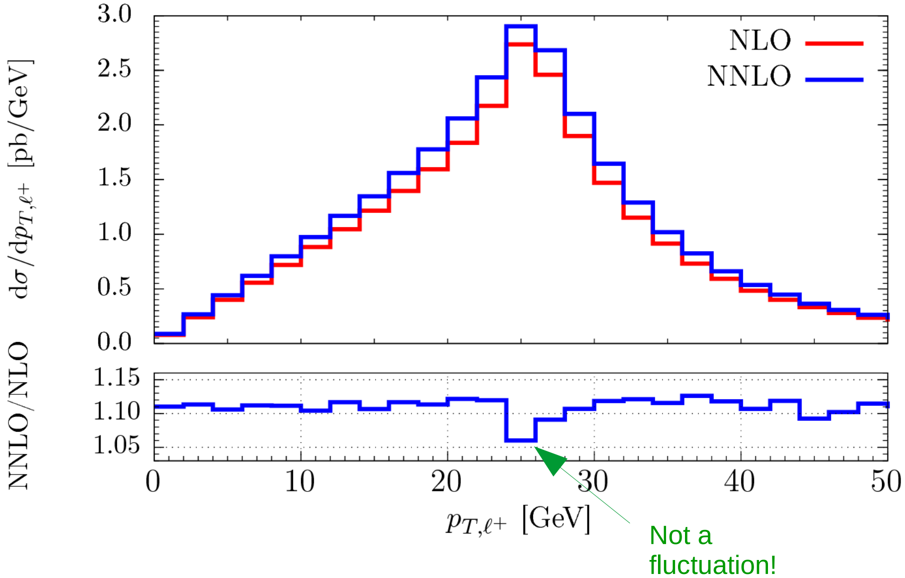
O(100 hours) runtime

- Lepton rapidity.
- O(10 hours): percent-level bin-to-bin fluctuations.
- O(100 hours): per-mille bin-to-bin fluctuations.

Differential distributions (II)



O(10 hours) runtime



O(100 hours) runtime

- Lepton transverse momentum.
- O(100 hours): percent-level bin-to-bin fluctuations.
- Delicate observable: receives contributions from large range of invariant masses.
  - Expected to improve once introduce *Z* boson propagator.
  - Competitive with state-of-the-art NNLO codes.

PROS

- Local.
- Clear origin of singularities.
- Explicit pole cancellation.
- Matrix elements can be evaluated in 4-dim.
- Process-independent: can handle colored final state colors, masses.

CONS

- Pole cancellation numerical:
  - Analytic cancellation not essential but would be nice.
  - Double-soft and triple collinear poles computed numerically.
  - Analytic integration possible, will require some effort.
- Not Lorentz invariant in intermediate steps:
  - Not a con *per se* but need to take care.
- Extension to colored final states and masses would require some work.
  - Concepts are all there, just need to be worked out before being implemented.

THANK YOU!