Loop amplitudes: The numerical approach

Stefan Weinzierl

Institut für Physik, Universität Mainz

I: **Local subtraction terms for loop amplitudes**

II: **Contour deformation**

III: **Cancellations at the integrand level**

(with UV divergences, non-zero spins and initial-state partons)

Numerical NLO QCD calculations

Use subtraction also for the virtual part:

$$\int_{n+1} d\sigma^{\mathrm{R}} + \int_{n} d\sigma^{\mathrm{V}} = \underbrace{\int_{n+1} \left(d\sigma^{\mathrm{R}} - d\sigma^{\mathrm{A}}_{\mathrm{R}}\right)}_{\text{convergent}} + \underbrace{\int_{n} (\mathbf{I} + \mathbf{L}) \otimes d\sigma^{B}}_{\text{finite}} + \underbrace{\int_{n+\text{loop}} \left(d\sigma^{\mathrm{V}} - d\sigma^{\mathrm{A}}_{\mathrm{V}}\right)}_{\text{convergent}}$$

- In the last term $d\sigma^{\mathrm{V}} - d\sigma^{\mathrm{A}}_{\mathrm{V}}$ the Monte Carlo integration is over a phase space integral of n final state particles plus a 4-dimensional loop integral.
- All explicit poles cancel in the combination $\mathbf{I} + \mathbf{L}$.
- Divergences of one-loop amplitudes related to IR-divergences (soft and collinear) and to UV-divergences.
- The IR-subtraction terms can be formulated at the level of amplitudes.

Z. Nagy, D. Soper, ’03; M. Assadsolimani, S. Becker, D. Götz, Ch. Reuschle, Ch. Schwan, S.W., ’09

Cancellations at the integrand level

$$\int_{n+1} d\sigma^{\mathrm{R}} + \int_{n} d\sigma^{\mathrm{V}} = \int_{n+1} \left(d\sigma^{\mathrm{R}} - d\sigma^{\mathrm{A}}_{\mathrm{R}}\right) + \underbrace{\int_{n} (\mathbf{I}+\mathbf{L}) \otimes d\sigma^{B}}_{\text{numerical integrable?}} + \int_{n+\mathrm{loop}} \left(d\sigma^{\mathrm{V}} - d\sigma^{\mathrm{A}}_{\mathrm{V}}\right)$$

- At NLO both $d\sigma^{\mathrm{A}}_{\mathrm{R}}$ and $d\sigma^{\mathrm{A}}_{\mathrm{V}}$ are easily integrated analytically.
- This is no longer true at NNLO and beyond.

$$\int_{n} (\mathbf{I}+\mathbf{L}) = \int_{n} \left[\int_{1} d\sigma^{\mathrm{A}}_{\mathrm{R}} + \int_{\mathrm{loop}} d\sigma^{\mathrm{A}}_{\mathrm{V}} + d\sigma^{\mathrm{V}}_{\mathrm{CT}} + d\sigma^{\mathrm{C}} \right].$$

- Unresolved phase space is $(D-1)$-dimensional.
- Loop momentum space is D-dimensional
- $d\sigma^{\mathrm{V}}_{\mathrm{CT}}$ counterterm from renormalisation
- $d\sigma^{\mathrm{C}}$ counterterm from factorisation

Loop-tree duality

A cyclic-ordered one-loop amplitude

$$A_n = \int \frac{d^D k}{(2\pi)^D} \frac{P(k)}{\prod_{j=1}^{n} \left(k_j^2 - m_j^2 + i\delta\right)}.$$

can be written with Cauchy's theorem as

$$A_n = -i \sum_{i=1}^{n} \int \frac{d^{D-1} k}{(2\pi)^{D-1}\, 2k_i^0} \frac{P(k)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \left[k_j^2 - m_j^2 - i\delta\left(k_j^0 - k_i^0\right)\right]} \Bigg|_{k_i^0=\sqrt{\vec{k}_i^2+m_i^2}},$$

Note the modified $i\delta$-prescription!

Catani, Gleisberg, Krauss, Rodrigo, Winter, '08

Maps

We need to relate the real unresolved phase space and the loop integration in the loop-tree duality approach:

Given a set $\{p_1, p_2, ..., p_n\}$ of external momenta and an on-shell loop momentum k there is an invertible map

$$\{p_1, p_2, ..., p_n\} \times \{k\} \quad \rightarrow \quad \{p'_1, p'_2, ..., p'_n, p'_{n+1}\}$$

Remark:

$$\{p'_1, p'_2, ..., p'_n, p'_{n+1}\} \quad \rightarrow \quad \{p_1, p_2, ..., p_n\}$$

is the standard Catani-Seymour projection.

Sborlini, Driencourt-Mangin, Hernandez-Pinto, German; Seth, S.W.

Collinear singularities

Problem with collinear singularities:

$d\sigma_{\mathrm{R}}^{\mathrm{A}}$: both partons have transverse polarisations, divergence in $g \to q\bar{q}$,

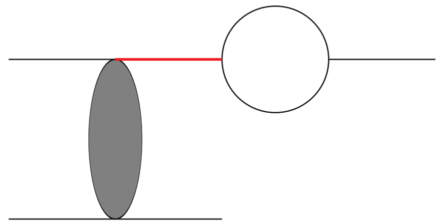
$d\sigma_{\mathrm{V}}^{\mathrm{A}}$: one parton has longitudinal polarisation, no divergence in $g \to q\bar{q}$.

Solution: Take field renormalisation constants into account:

$$
\begin{aligned}
Z_2 &= 1 = 1 + \frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\varepsilon_{\mathrm{IR}}} - \frac{1}{\varepsilon_{\mathrm{UV}}} \right) \\
Z_3 &= 1 = 1 + \frac{\alpha_s}{4\pi} (2C_A - \beta_0) \left(\frac{1}{\varepsilon_{\mathrm{IR}}} - \frac{1}{\varepsilon_{\mathrm{UV}}} \right)
\end{aligned}
$$

Field renormalisation

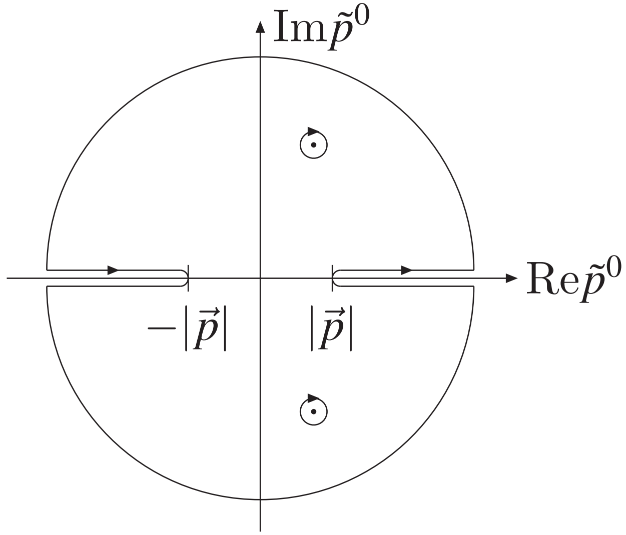
Field renormalisation constants derived from self-energies.



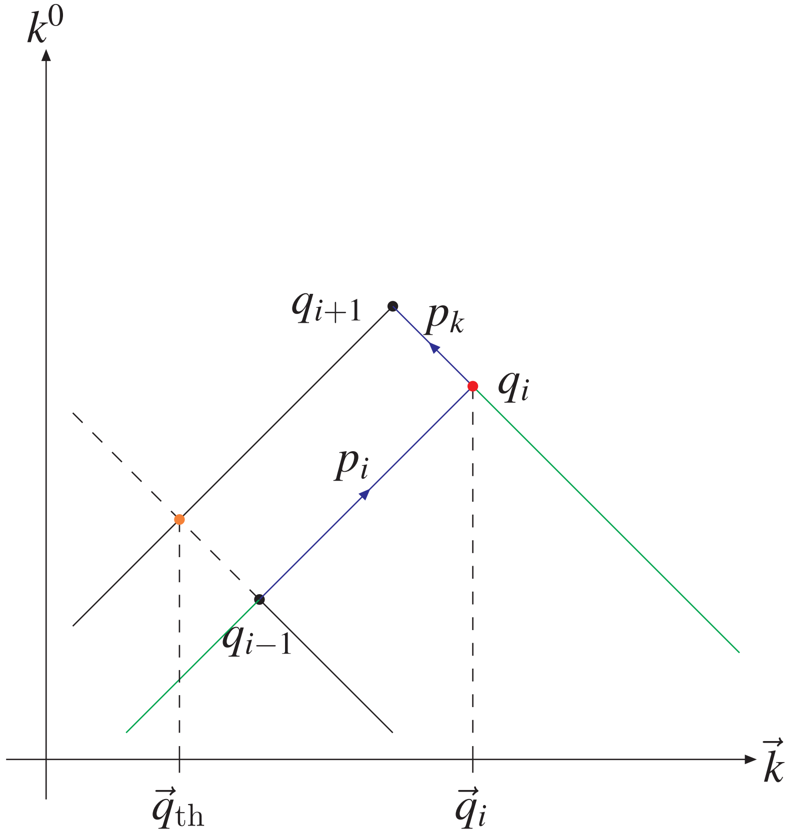
Problem: Internal on-shell propagator.

Solution: Use dispersion relation.

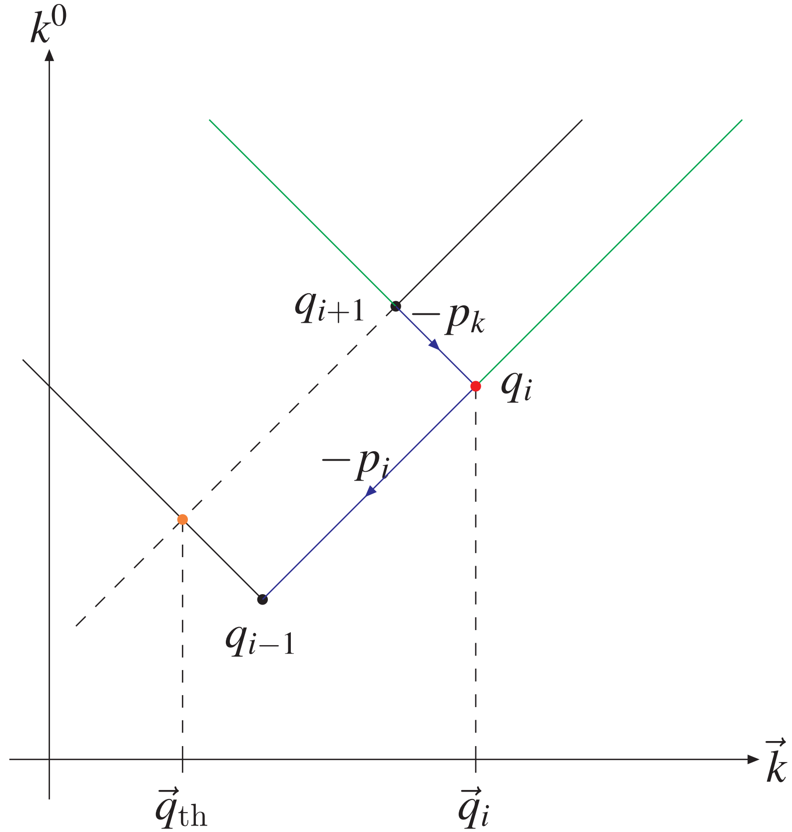
Soper, '01; Seth, S.W., '16



Final state singularities



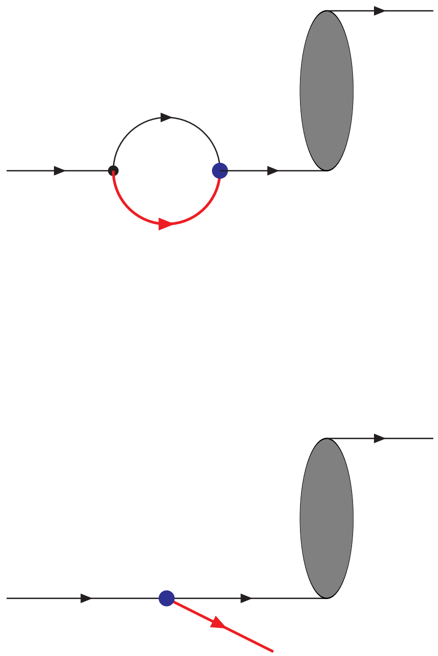
emitter i, spectator k

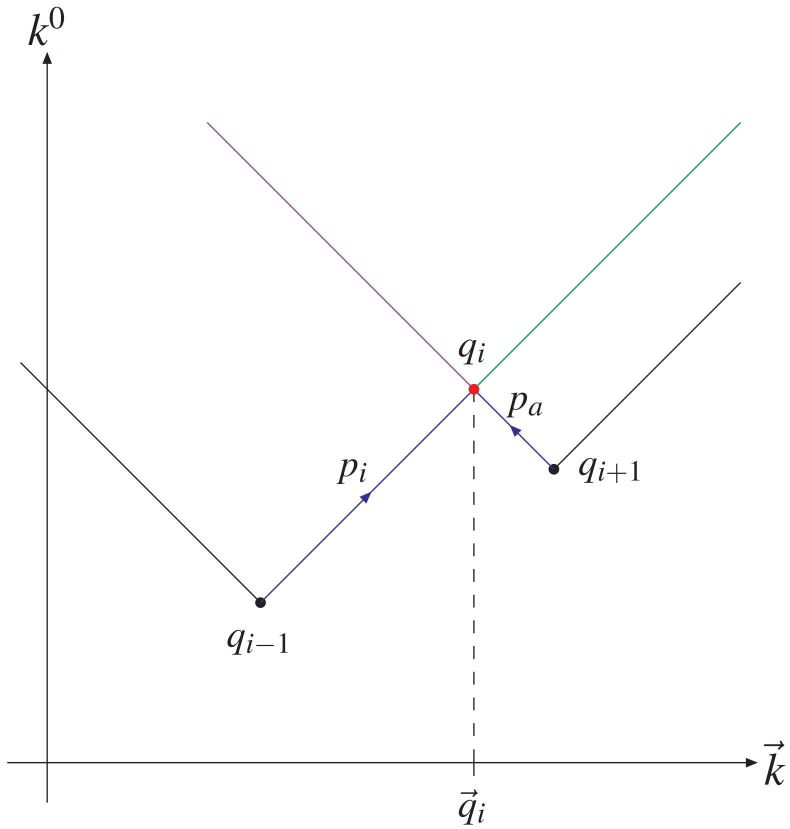


emitter k, spectator i

Initial-state collinear singularities

Problem: For initial-state collinear singularities the regions do not match.





Initial-state collinear singularities

We still have to include the counterterm from factorisation.

$$d\sigma^{\mathrm{C}} = \frac{\alpha_s}{4\pi} \int_0^1 dx_a \, \frac{2}{\varepsilon} \left(\frac{\mu_F^2}{\mu^2} \right)^{-\varepsilon} P^{a'a}(x_a) \, d\sigma^{\mathrm{B}}(\ldots, x_a p'_a, \ldots) \, .$$

Example of splitting function:

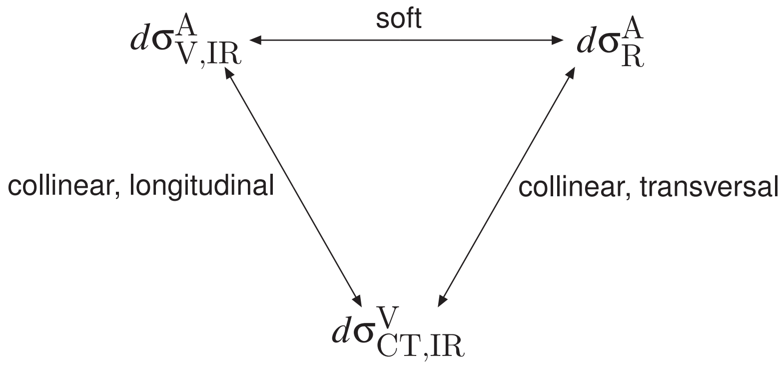
$$P^{gg} = 2C_A \left[\left. \frac{1}{1-x} \right|_+ + \frac{1-x}{x} - 1 + x(1-x) \right] + \frac{\beta_0}{2} \delta(1-x) \, .$$

Solution: Unintegrated representation of the collinear subtraction term $d\sigma^{\mathrm{C}}$.

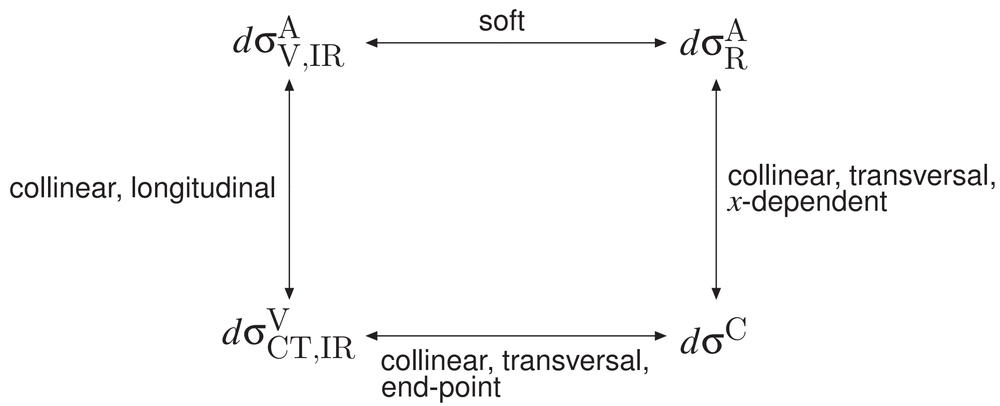
- x-dependent part matches on real contribution
- end-point part matches on virtual contribution

Cancellations of infrared singularities

Only final-state particles:



With initial-state particles:



Comment on remaining analytic integrals

Does the numerical approach eliminate the need of any analytic calculation of an integral?

- No analytic integral required where divergences cancel (i.e. final-state soft or collinear)
- But: UV divergences removed by renormalisation, initial-state collinear divergences by factorisation, this introduces a scheme dependence.
- Have to reproduce the finite terms associated to a given renormalisation scheme / factorisation scheme ($\overline{\text{MS}}$-scheme,...)
- Need simple integrals analytically

$$\text{Renormalisation}: \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m)^\nu}, \qquad \text{Factorisation}: \int_0^1 dx\, x^{\nu-\varepsilon} (1-x)^{-\varepsilon}.$$